Calculus and Analytical Geometry

Complete Course Notes

BS/ADP CS-IT-SE

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Chapter 1

Limits and Continuity

Introduction to Functions

Functions are fundamental building blocks of calculus. They describe relationships between quantities and allow us to model real-world phenomena mathematically. In this section, we establish the foundational concepts that will be used throughout the course.

Definition 1.1: Function

A **function** f from a set A to a set B is a rule that assigns to each element x in A exactly one element y in B. We write:

$$f : A \to B \quad \text{or} \quad y = f(x)$$

Key Points:

- x is called the **independent variable** or **input**
- y is called the **dependent variable** or **output**
- The set A is called the **domain** of f
- The set of all possible outputs is called the **range** of f
- Each input must produce exactly ONE output (uniqueness)

Example 1: Identifying Functions

Problem: Determine which of the following relations are functions:
(a) $f(x) = x^2 + 1$
(b) $x^2 + y^2 = 25$ (circle equation)
(c) $g(t) = \sqrt{t - 1}$

Solution:
(a) For $f(x) = x^2 + 1$:
For any value of x, we compute x^2, then add 1. This gives exactly one output.

$$\begin{aligned} x = 2 &\implies f(2) = 2^2 + 1 = 5 \\ x = -3 &\implies f(-3) = (-3)^2 + 1 = 10 \end{aligned}$$

✓ **This IS a function.**

(b) For $x^2 + y^2 = 25$:
If $x = 0$, then $y^2 = 25$, which gives $y = 5$ or $y = -5$ (two outputs!).
× **This is NOT a function** because one input produces multiple outputs.

(c) For $g(t) = \sqrt{t-1}$:
For any valid t, the square root produces exactly one non-negative value.

$$t = 5 \implies g(5) = \sqrt{5-1} = \sqrt{4} = 2$$

✓ **This IS a function.**
⚠ **Note:** The domain is $t \geq 1$ (we cannot take square root of negative numbers in real numbers).

Definition 1.2: Domain and Range

For a function $f : A \to B$:
Domain: The set of all permissible input values for which the function is defined.

$$\text{Domain}(f) = \{x : f(x) \text{ exists}\}$$

Range: The set of all actual output values produced by the function.

$$\text{Range}(f) = \{y : y = f(x) \text{ for some } x \in \text{Domain}(f)\}$$

ℹ Common Domain Restrictions:

- **Division:** Cannot divide by zero $\implies$ exclude values where denominator $= 0$
- **Square roots:** Expression under $\sqrt{\ }$ must be non-negative (for real functions)
- **Logarithms:** Argument of $\ln(x)$ or $\log(x)$ must be positive

Example 2: Finding Domain

Problem: Find the domain of the following functions:
(a) $f(x) = \dfrac{1}{x^2 - x}$
(b) $h(x) = \sqrt{x+2}$
(c) $g(x) = \dfrac{1}{x^2 - 3x}$

Solution:
(a) For $f(x) = \dfrac{1}{x^2 - x}$:
We need $x^2 - x \neq 0$ (denominator cannot be zero).
Factor: $x(x-1) \neq 0$
This means $x \neq 0$ and $x \neq 1$.
Domain: $(-\infty, 0) \cup (0, 1) \cup (1, \infty)$ or $\mathbb{R} \setminus \{0, 1\}$

(b) For $h(x) = \sqrt{x+2}$:
We need $x + 2 \geq 0$ (expression under square root must be non-negative).
Solving: $x \geq -2$
Domain: $[-2, \infty)$

(c) For $g(x) = \dfrac{1}{x^2 - 3x}$:
We need $x^2 - 3x \neq 0$.
Factor: $x(x-3) \neq 0$
This means $x \neq 0$ and $x \neq 3$.
Domain: $\mathbb{R} \setminus \{0, 3\}$ or $(-\infty, 0) \cup (0, 3) \cup (3, \infty)$
Verification: Test $x = 1$: $g(1) = \dfrac{1}{1-3} = -\dfrac{1}{2}$ ✓ (defined)

Definition 1.3: Even and Odd Functions

A function f is called:

Even if $f(-x) = f(x)$ for all x in the domain.

- Graph is symmetric about the y-axis
- Example: $f(x) = x^2$, $f(x) = \cos x$

Odd if $f(-x) = -f(x)$ for all x in the domain.

- Graph is symmetric about the origin
- Example: $f(x) = x^3$, $f(x) = \sin x$

⚠ **Note:** A function can be neither even nor odd!

Example 3: Determining Even or Odd Functions

Problem: Determine whether the following functions are even, odd, or neither:
(a) $f(x) = x^5 + x$
(b) $g(x) = x^4 - 2x^2 + 1$
(c) $h(x) = x^3 + x^2$

Solution:
(a) For $f(x) = x^5 + x$:
Compute $f(-x)$:

$$\begin{aligned} f(-x) &= (-x)^5 + (-x) \\ &= -x^5 - x \\ &= -(x^5 + x) \\ &= -f(x) \end{aligned}$$

Since $f(-x) = -f(x)$, the function is **ODD**. ✓

(b) For $g(x) = x^4 - 2x^2 + 1$:
Compute $g(-x)$:

$$\begin{aligned} g(-x) &= (-x)^4 - 2(-x)^2 + 1 \\ &= x^4 - 2x^2 + 1 \\ &= g(x) \end{aligned}$$

Since $g(-x) = g(x)$, the function is **EVEN**. ✓

(c) For $h(x) = x^3 + x^2$:
Compute $h(-x)$:

$$\begin{aligned} h(-x) &= (-x)^3 + (-x)^2 \\ &= -x^3 + x^2 \end{aligned}$$

Now check:

- Is $h(-x) = h(x)$? Compare $-x^3 + x^2$ with $x^3 + x^2$ × No
- Is $h(-x) = -h(x)$? Compare $-x^3 + x^2$ with $-(x^3 + x^2) = -x^3 - x^2$ × No

The function is **NEITHER even nor odd**.

Definition 1.4: Composition of Functions

If f and g are functions, the **composition** of f with g, denoted $(f \circ g)$, is defined by:

$$(f \circ g)(x) = f(g(x))$$

This means: first apply g to x, then apply f to the result.

Domain of Composition:

The domain of $f \circ g$ consists of all x in the domain of g such that $g(x)$ is in the domain of f.

Important: In general, $f \circ g \neq g \circ f$ (composition is NOT commutative!)

Example 4: Function Composition

Problem: Let $f(x) = x^2 + 1$ and $g(x) = 2x - 3$. Find:
(a) $(f \circ g)(x)$
(b) $(g \circ f)(x)$
(c) Evaluate $(f \circ g)(2)$

Solution:
(a) Finding $(f \circ g)(x) = f(g(x))$:
First find $g(x)$: $g(x) = 2x - 3$
Now apply f to this result:

$$\begin{aligned} f(g(x)) &= f(2x - 3) \\ &= (2x - 3)^2 + 1 \\ &= 4x^2 - 12x + 9 + 1 \\ &= 4x^2 - 12x + 10 \end{aligned}$$

Answer: $(f \circ g)(x) = 4x^2 - 12x + 10$

(b) Finding $(g \circ f)(x) = g(f(x))$:
First find $f(x)$: $f(x) = x^2 + 1$

Now apply g to this result:

$$\begin{aligned} g(f(x)) &= g(x^2+1) \\ &= 2(x^2+1)-3 \\ &= 2x^2+2-3 \\ &= 2x^2-1 \end{aligned}$$

Answer: $(g \circ f)(x) = 2x^2 - 1$

⚠ **Observation:** Note that $f \circ g \neq g \circ f$! This confirms composition is not commutative.

(c) Evaluating $(f \circ g)(2)$:
Method 1 (direct substitution in composition):

$$(f \circ g)(2) = 4(2)^2 - 12(2) + 10 = 16 - 24 + 10 = 2$$

Method 2 (step by step):

$$\begin{aligned} g(2) &= 2(2) - 3 = 1 \\ f(g(2)) &= f(1) = 1^2 + 1 = 2 \end{aligned}$$

Answer: $\boxed{(f \circ g)(2) = 2}$

Definition 1.5: Inverse Function

A function f has an **inverse function** f^{-1} if there exists a function f^{-1} such that:

$$f^{-1}(f(x)) = x \quad \text{for all } x \text{ in domain of } f$$

$$f(f^{-1}(y)) = y \quad \text{for all } y \text{ in domain of } f^{-1}$$

ℹ **Key Properties:**

- A function has an inverse if and only if it is **one-to-one** (each output corresponds to exactly one input)
- The graph of f^{-1} is the reflection of the graph of f across the line $y = x$
- Domain of f^{-1} = Range of f
- Range of f^{-1} = Domain of f

⚠ **Warning:** $f^{-1}(x) \neq \dfrac{1}{f(x)}$! The notation means inverse, not reciprocal.

Example 5: Finding Inverse Functions

Problem: Find the inverse function $f^{-1}(x)$ for:
(a) $f(x) = 2x + 3$
(b) $f(x) = x^3 + 1$

Solution:

(a) For $f(x) = 2x + 3$:
Step 1: Write $y = f(x)$

$$y = 2x + 3$$

Step 2: Solve for x in terms of y

$$\begin{aligned} y &= 2x + 3 \\ y - 3 &= 2x \\ x &= \frac{y-3}{2} \end{aligned}$$

Step 3: Replace y with x to get $f^{-1}(x)$

$$f^{-1}(x) = \frac{x-3}{2}$$

Verification: Check that $f(f^{-1}(x)) = x$

$$\begin{aligned} f(f^{-1}(x)) &= f\left(\frac{x-3}{2}\right) \\ &= 2\left(\frac{x-3}{2}\right) + 3 \\ &= x - 3 + 3 \\ &= x \quad \checkmark \end{aligned}$$

Answer: $\boxed{f^{-1}(x) = \frac{x-3}{2}}$

(b) For $f(x) = x^3 + 1$:
Step 1: Write $y = f(x)$

$$y = x^3 + 1$$

Step 2: Solve for x in terms of y

$$\begin{aligned} y &= x^3 + 1 \\ y - 1 &= x^3 \\ x &= \sqrt[3]{y-1} \end{aligned}$$

Step 3: Replace y with x

$$f^{-1}(x) = \sqrt[3]{x-1}$$

Verification:

$$\begin{aligned} f(f^{-1}(x)) &= f(\sqrt[3]{x-1}) \\ &= (\sqrt[3]{x-1})^3 + 1 \\ &= x - 1 + 1 \\ &= x \quad \checkmark \end{aligned}$$

Answer: $\boxed{f^{-1}(x) = \sqrt[3]{x-1}}$

Section 1.1 Summary

Concept	Key Formula/Property
Function Definition	Each input x produces exactly ONE output $f(x)$
Domain	Set of all valid inputs: $\{x : f(x) \text{ exists}\}$
Range	Set of all actual outputs: $\{y : y = f(x)\}$
Even Function	$f(-x) = f(x)$ (symmetric about y-axis)
Odd Function	$f(-x) = -f(x)$ (symmetric about origin)
Composition	$(f \circ g)(x) = f(g(x))$
Inverse Function	$f^{-1}(f(x)) = x$ and $f(f^{-1}(x)) = x$

Problem-Solving Tips:

- Always identify domain restrictions FIRST (division by zero, square roots, logarithms)
- To check even/odd: substitute $-x$ and compare with $f(x)$ and $-f(x)$
- For composition: work from inside out
- To find inverse: swap x and y, then solve for y
- Always verify your inverse by checking $f(f^{-1}(x)) = x$

Introduction to Limits

The concept of a limit is the foundation of calculus. Limits allow us to describe the behavior of functions as inputs approach certain values, even when the function may not be defined at those exact points. This section introduces limits intuitively before we develop formal techniques.

Definition 1.6: Informal Limit Definition

We write:

$$\lim_{x \to a} f(x) = L$$

and say "*the limit of $f(x)$ as x approaches a is L*" if we can make the values of $f(x)$ arbitrarily close to L by taking x sufficiently close to a (but not equal to a).

Key Understanding:

- The limit describes behavior **near** a, not necessarily **at** a
- $f(a)$ does not need to exist for the limit to exist
- $f(a)$ does not need to equal L for the limit to exist
- We approach a from both left and right sides

Important Distinction:

$$\lim_{x \to a} f(x) = L \quad \text{vs.} \quad f(a) = L$$

These are **different statements**! The limit is about approaching, not arriving.

Example 6: Understanding Limits Graphically

Problem: Consider the function:

$$f(x) = \frac{x^2 - 4}{x - 2}$$

(a) What is $f(2)$?
(b) What is $\lim_{x \to 2} f(x)$?

Solution:
(a) Finding $f(2)$:
Direct substitution:

$$f(2) = \frac{2^2 - 4}{2 - 2} = \frac{0}{0}$$

This is **undefined**! So $f(2)$ does not exist.

(b) Finding $\lim_{x \to 2} f(x)$:
Even though $f(2)$ doesn't exist, we can examine what happens as x gets close to 2.
First, simplify the function:

$$\begin{aligned} f(x) &= \frac{x^2 - 4}{x - 2} \\ &= \frac{(x-2)(x+2)}{x - 2} \\ &= x + 2 \quad (\text{for } x \neq 2) \end{aligned}$$

Now create a table of values approaching $x = 2$:

x (from left)	$f(x) = x + 2$	x (from right)	$f(x) = x + 2$
1.9	3.9	2.1	4.1
1.99	3.99	2.01	4.01
1.999	3.999	2.001	4.001
1.9999	3.9999	2.0001	4.0001

Observation: As x approaches 2 from either side, $f(x)$ approaches 4.
Answer: $\lim_{x\to 2} f(x) = 4$, even though $f(2)$ is undefined!
⚠ **Key Insight:** The limit exists and equals 4, but the function has a "hole" at $x = 2$.

Definition 1.7: One-Sided Limits

Left-hand limit:

$$\lim_{x\to a^-} f(x) = L$$

means $f(x)$ approaches L as x approaches a from the left (values less than a).
Right-hand limit:

$$\lim_{x\to a^+} f(x) = L$$

means $f(x)$ approaches L as x approaches a from the right (values greater than a).

Existence of Limit:
The (two-sided) limit $\lim_{x\to a} f(x) = L$ exists if and only if:

$$\lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x) = L$$

Both one-sided limits must exist and be equal!

Example 7: One-Sided Limits

Problem: Consider the piecewise function:

$$f(x) = \begin{cases} x+2, & x \le -1 \\ c+2, & x > -1 \end{cases}$$

Find the value of c so that $\lim_{x\to -1} f(x)$ exists.

Solution:
For the limit to exist at $x = -1$, we need:

$$\lim_{x\to -1^-} f(x) = \lim_{x\to -1^+} f(x)$$

Step 1: Find the left-hand limit
As x approaches -1 from the left, we use the first piece: $f(x) = x + 2$

$$\begin{aligned} \lim_{x\to -1^-} f(x) &= \lim_{x\to -1^-} (x+2) \\ &= -1 + 2 \\ &= 1 \end{aligned}$$

Step 2: Find the right-hand limit
As x approaches -1 from the right, we use the second piece: $f(x) = c + 2$

$$\begin{aligned}\lim_{x \to -1^+} f(x) &= \lim_{x \to -1^+} (c + 2) \\ &= c + 2\end{aligned}$$

Step 3: Set them equal
For the limit to exist:

$$\begin{aligned}\lim_{x \to -1^-} f(x) &= \lim_{x \to -1^+} f(x) \\ 1 &= c + 2 \\ c &= -1\end{aligned}$$

Answer: $\boxed{c = -1}$
Verification: With $c = -1$:

- $\lim_{x \to -1^-} f(x) = 1$
- $\lim_{x \to -1^+} f(x) = -1 + 2 = 1$
- Both equal, so $\lim_{x \to -1} f(x) = 1$ exists! ✓

Example 8: Evaluating Simple Limits

Problem: Evaluate the following limits:
(a) $\lim_{x \to 3} (2x + 5)$
(b) $\lim_{x \to -2} (x^2 - 3x + 1)$
(c) $\lim_{x \to 1} \dfrac{x^2 + x - 2}{x^2 - x}$

Solution:
(a) For $\lim_{x \to 3} (2x + 5)$:
This is a polynomial, so we can substitute directly:

$$\begin{aligned}\lim_{x \to 3} (2x + 5) &= 2(3) + 5 \\ &= 6 + 5 \\ &= 11\end{aligned}$$

Answer: $\boxed{11}$

(b) For $\lim_{x \to -2} (x^2 - 3x + 1)$:
Again, polynomial, so direct substitution:

$$\begin{aligned}\lim_{x \to -2} (x^2 - 3x + 1) &= (-2)^2 - 3(-2) + 1 \\ &= 4 + 6 + 1 \\ &= 11\end{aligned}$$

Answer: $\boxed{11}$

(c) For $\lim_{x\to 1} \frac{x^2+x-2}{x^2-x}$:
First try direct substitution:

$$\frac{1^2+1-2}{1^2-1} = \frac{0}{0}$$

This is indeterminate! We need to simplify first.
Factor numerator and denominator:

$$\text{Numerator: } x^2+x-2 = (x+2)(x-1)$$
$$\text{Denominator: } x^2-x = x(x-1)$$

So:

$$\begin{aligned} \lim_{x\to 1} \frac{x^2+x-2}{x^2-x} &= \lim_{x\to 1} \frac{(x+2)(x-1)}{x(x-1)} \\ &= \lim_{x\to 1} \frac{x+2}{x} \quad \text{(cancel } x-1\text{, valid for } x \neq 1) \\ &= \frac{1+2}{1} \\ &= 3 \end{aligned}$$

Answer: $\boxed{3}$
Strategy: When you get $\frac{0}{0}$, factor and cancel common terms!

Definition 1.8: Basic Limit Laws

If $\lim_{x\to a} f(x) = L$ and $\lim_{x\to a} g(x) = M$, then:

1. **Sum Rule:**
$$\lim_{x\to a}[f(x)+g(x)] = L+M$$
2. **Difference Rule:**
$$\lim_{x\to a}[f(x)-g(x)] = L-M$$
3. **Constant Multiple Rule:**
$$\lim_{x\to a}[c\cdot f(x)] = c\cdot L$$
4. **Product Rule:**
$$\lim_{x\to a}[f(x)\cdot g(x)] = L\cdot M$$
5. **Quotient Rule:**
$$\lim_{x\to a}\frac{f(x)}{g(x)} = \frac{L}{M}, \quad \text{provided } M \neq 0$$
6. **Power Rule:**
$$\lim_{x\to a}[f(x)]^n = L^n$$

Requirement: All these laws require that the individual limits exist!

Example 9: Using Limit Laws

Problem: Given that $\lim_{x\to 2} f(x) = 5$ and $\lim_{x\to 2} g(x) = -3$, find:

(a) $\lim_{x\to 2}[f(x) + 2g(x)]$

(b) $\lim_{x\to 2}[f(x) \cdot g(x)]$

(c) $\lim_{x\to 2} \dfrac{f(x)}{g(x)+5}$

Solution:

(a) Using Sum Rule and Constant Multiple Rule:

$$\begin{aligned}\lim_{x\to 2}[f(x) + 2g(x)] &= \lim_{x\to 2} f(x) + \lim_{x\to 2}[2g(x)] \\ &= \lim_{x\to 2} f(x) + 2\lim_{x\to 2} g(x) \\ &= 5 + 2(-3) \\ &= 5 - 6 \\ &= -1\end{aligned}$$

Answer: $\boxed{-1}$

(b) Using Product Rule:

$$\begin{aligned}\lim_{x\to 2}[f(x) \cdot g(x)] &= \left[\lim_{x\to 2} f(x)\right] \cdot \left[\lim_{x\to 2} g(x)\right] \\ &= 5 \cdot (-3) \\ &= -15\end{aligned}$$

Answer: $\boxed{-15}$

(c) Using Quotient Rule:
First find the limit of the denominator:

$$\begin{aligned}\lim_{x\to 2}[g(x) + 5] &= \lim_{x\to 2} g(x) + 5 \\ &= -3 + 5 \\ &= 2 \neq 0 \quad \checkmark\end{aligned}$$

Now apply quotient rule:

$$\begin{aligned}\lim_{x\to 2} \frac{f(x)}{g(x)+5} &= \frac{\lim_{x\to 2} f(x)}{\lim_{x\to 2}[g(x)+5]} \\ &= \frac{5}{2}\end{aligned}$$

Answer: $\boxed{\frac{5}{2}}$

Example 10: Limit of a Polynomial

Problem: Evaluate:

$$\lim_{x\to-3}\sqrt{\frac{x+2}{x+1}}$$

Solution:

Step 1: Check if direct substitution works

First verify the expression is defined at $x=-3$:

$$\frac{-3+2}{-3+1}=\frac{-1}{-2}=\frac{1}{2}>0 \quad \checkmark$$

Since we're taking square root of a positive number, we can proceed.

Step 2: Apply limit laws

Using the composition property (limit of square root equals square root of limit):

$$\begin{aligned}
\lim_{x\to-3}\sqrt{\frac{x+2}{x+1}} &= \sqrt{\lim_{x\to-3}\frac{x+2}{x+1}} \\
&= \sqrt{\frac{\lim_{x\to-3}(x+2)}{\lim_{x\to-3}(x+1)}} \quad \text{(quotient rule)} \\
&= \sqrt{\frac{-3+2}{-3+1}} \\
&= \sqrt{\frac{-1}{-2}} \\
&= \sqrt{\frac{1}{2}} \\
&= \frac{1}{\sqrt{2}} \\
&= \frac{\sqrt{2}}{2}
\end{aligned}$$

Answer: $\boxed{\frac{\sqrt{2}}{2}}$

Section 1.2 Summary

Concept	Meaning
$\lim_{x\to a} f(x)=L$	$f(x)$ approaches L as x approaches a
Limit vs Function Value	$\lim_{x\to a} f(x)$ can exist even if $f(a)$ doesn't!
One-Sided Limits	Left: $x\to a^-$; Right: $x\to a^+$
Limit Existence	Limit exists $\iff$ both one-sided limits exist and are equal
Direct Substitution	Works for polynomials and continuous functions
Indeterminate Form $\frac{0}{0}$	Factor and cancel, then try again

Problem-Solving Strategy:

1. Try direct substitution first

2. If you get a number (not $\frac{0}{0}$), that's your answer
3. If you get $\frac{0}{0}$, factor numerator and denominator
4. Cancel common factors and try substitution again
5. For piecewise functions, check one-sided limits separately

⚠ Common Mistakes to Avoid:

- Confusing $\lim_{x \to a} f(x)$ with $f(a)$
- Forgetting to check both one-sided limits for piecewise functions
- Canceling terms incorrectly when dealing with $\frac{0}{0}$
- Dividing by zero in quotient rule (denominator limit must be non-zero!)

Techniques of Finding Limits

In the previous section, we introduced limits informally and used direct substitution for simple cases. Now we develop systematic techniques for evaluating limits, especially when direct substitution fails or produces indeterminate forms.

Definition 1.9: Standard Limit Techniques

When evaluating $\lim_{x \to a} f(x)$, use these techniques in order:

Technique 1: Direct Substitution

- If f is continuous at a, then $\lim_{x \to a} f(x) = f(a)$
- Works for polynomials, rational functions (where denominator $\neq 0$)

Technique 2: Factoring and Canceling

- Used when direct substitution gives $\frac{0}{0}$
- Factor both numerator and denominator
- Cancel common factors
- Try direct substitution again

Technique 3: Rationalization

- Used when square roots appear in numerator or denominator
- Multiply by conjugate: $(a - b)(a + b) = a^2 - b^2$

Technique 4: Common Denominator

- Used for complex fractions
- Combine fractions with LCD
- Simplify and apply other techniques

Example 11: Factoring Technique

Problem: Evaluate:

$$\lim_{x \to 2} \frac{4x - x^2}{2 - \sqrt{x}}$$

Solution:

Step 1: Try direct substitution

$$\frac{4(2) - 2^2}{2 - \sqrt{2}} = \frac{8 - 4}{2 - \sqrt{2}} = \frac{4}{2 - \sqrt{2}}$$

This is defined (not $\frac{0}{0}$), so we can compute:

$$\frac{4}{2 - \sqrt{2}} = \frac{4}{2 - 1.414...} \approx \frac{4}{0.586} \approx 6.83$$

But let's rationalize the denominator for an exact answer.

Step 2: Rationalize the denominator
Multiply numerator and denominator by the conjugate $(2+\sqrt{x})$:

$$\begin{aligned}\lim_{x\to 2}\frac{4x-x^2}{2-\sqrt{x}} &= \lim_{x\to 2}\frac{4x-x^2}{2-\sqrt{x}}\cdot\frac{2+\sqrt{x}}{2+\sqrt{x}}\\ &= \lim_{x\to 2}\frac{(4x-x^2)(2+\sqrt{x})}{(2-\sqrt{x})(2+\sqrt{x})}\\ &= \lim_{x\to 2}\frac{(4x-x^2)(2+\sqrt{x})}{4-x}\end{aligned}$$

Step 3: Factor the numerator
Factor $4x-x^2 = x(4-x) = -x(x-4)$:

$$\begin{aligned}&= \lim_{x\to 2}\frac{-x(x-4)(2+\sqrt{x})}{4-x}\\ &= \lim_{x\to 2}\frac{-x(x-4)(2+\sqrt{x})}{-(x-4)}\\ &= \lim_{x\to 2}\frac{x(x-4)(2+\sqrt{x})}{x-4}\\ &= \lim_{x\to 2} x(2+\sqrt{x}) \quad \text{(cancel } x-4\text{)}\end{aligned}$$

Step 4: Apply direct substitution

$$\begin{aligned}&= 2(2+\sqrt{2})\\ &= 4+2\sqrt{2}\end{aligned}$$

Answer: $\boxed{4+2\sqrt{2}}$
Verification: $4+2\sqrt{2}\approx 4+2(1.414) = 6.828$ ✓ matches our estimate!

Example 12: Rationalization with Square Roots

Problem: Evaluate:

$$\lim_{x\to 5}\frac{4-x}{\sqrt{x+4}-\sqrt{x-4}}$$

Solution:
Step 1: Try direct substitution

$$\frac{4-5}{\sqrt{5+4}-\sqrt{5-4}} = \frac{-1}{\sqrt{9}-\sqrt{1}} = \frac{-1}{3-1} = \frac{-1}{2}$$

This works! But let's verify using algebraic manipulation.
Alternative Method - Rationalization:
Multiply by conjugate of denominator: $(\sqrt{x+4}+\sqrt{x-4})$

$$
\begin{aligned}
&= \lim_{x\to 5} \frac{4-x}{\sqrt{x+4}-\sqrt{x-4}} \cdot \frac{\sqrt{x+4}+\sqrt{x-4}}{\sqrt{x+4}+\sqrt{x-4}} \\
&= \lim_{x\to 5} \frac{(4-x)(\sqrt{x+4}+\sqrt{x-4})}{(\sqrt{x+4})^2-(\sqrt{x-4})^2} \\
&= \lim_{x\to 5} \frac{(4-x)(\sqrt{x+4}+\sqrt{x-4})}{(x+4)-(x-4)} \\
&= \lim_{x\to 5} \frac{(4-x)(\sqrt{x+4}+\sqrt{x-4})}{8} \\
&= \frac{1}{8}\lim_{x\to 5}(4-x)(\sqrt{x+4}+\sqrt{x-4})
\end{aligned}
$$

Step 2: Substitute $x = 5$

$$
\begin{aligned}
&= \frac{1}{8}(4-5)(\sqrt{5+4}+\sqrt{5-4}) \\
&= \frac{1}{8}(-1)(\sqrt{9}+\sqrt{1}) \\
&= \frac{1}{8}(-1)(3+1) \\
&= \frac{-4}{8} \\
&= -\frac{1}{2}
\end{aligned}
$$

Answer: $\boxed{-\frac{1}{2}}$

Both methods agree! ✓

Example 13: Rationalization Technique (Classic Type)

Problem: Evaluate:

$$\lim_{x\to 9} \frac{\sqrt{x}-3}{x-9}$$

Solution:

Step 1: Try direct substitution

$$\frac{\sqrt{9}-3}{9-9} = \frac{3-3}{0} = \frac{0}{0}$$

Indeterminate form! Need to simplify.

Step 2: Rationalize the numerator

Multiply by conjugate $(\sqrt{x}+3)$:

$$
\begin{aligned}
\lim_{x\to 9} \frac{\sqrt{x}-3}{x-9} &= \lim_{x\to 9} \frac{\sqrt{x}-3}{x-9} \cdot \frac{\sqrt{x}+3}{\sqrt{x}+3} \\
&= \lim_{x\to 9} \frac{(\sqrt{x})^2-3^2}{(x-9)(\sqrt{x}+3)} \\
&= \lim_{x\to 9} \frac{x-9}{(x-9)(\sqrt{x}+3)}
\end{aligned}
$$

Step 3: Cancel common factor $(x-9)$

$$= \lim_{x\to 9} \frac{1}{\sqrt{x}+3}$$

Step 4: Apply direct substitution

$$= \frac{1}{\sqrt{9}+3}$$
$$= \frac{1}{3+3}$$
$$= \frac{1}{6}$$

Answer: $\boxed{\frac{1}{6}}$

⚠ **Key Pattern:** When you see $\sqrt{x}-a$ in numerator and $x-a^2$ in denominator, rationalize!

Example 14: Complex Fraction Technique

Problem: Evaluate:

$$\lim_{h\to 0} \frac{\frac{1}{x+h}-\frac{1}{x}}{h}$$

(This is the derivative of $\frac{1}{x}$ from first principles!)

Solution:

Step 1: Simplify the complex fraction

Find common denominator for the numerator:

$$\begin{aligned} \frac{1}{x+h}-\frac{1}{x} &= \frac{x}{x(x+h)}-\frac{x+h}{x(x+h)} \\ &= \frac{x-(x+h)}{x(x+h)} \\ &= \frac{x-x-h}{x(x+h)} \\ &= \frac{-h}{x(x+h)} \end{aligned}$$

Step 2: Substitute back into the limit

$$\begin{aligned} \lim_{h\to 0} \frac{\frac{1}{x+h}-\frac{1}{x}}{h} &= \lim_{h\to 0} \frac{\frac{-h}{x(x+h)}}{h} \\ &= \lim_{h\to 0} \frac{-h}{x(x+h)} \cdot \frac{1}{h} \\ &= \lim_{h\to 0} \frac{-h}{hx(x+h)} \\ &= \lim_{h\to 0} \frac{-1}{x(x+h)} \quad \text{(cancel } h\text{)} \end{aligned}$$

Step 3: Apply direct substitution

$$
\begin{aligned}
&= \frac{-1}{x(x+0)} \\
&= \frac{-1}{x^2} \\
&= -\frac{1}{x^2}
\end{aligned}
$$

Answer: $\boxed{-\frac{1}{x^2}}$

Insight: This confirms that $\frac{d}{dx}\left(\frac{1}{x}\right) = -\frac{1}{x^2}$!

Example 15: Factoring Polynomial Limits

Problem: Evaluate:

$$\lim_{x\to 2} \frac{x^3 + 2x - 12}{5 - x}$$

Solution:

Step 1: Try direct substitution

$$
\begin{aligned}
\frac{2^3 + 2(2) - 12}{5 - 2} &= \frac{8 + 4 - 12}{3} \\
&= \frac{0}{3} \\
&= 0
\end{aligned}
$$

This is well-defined (not $\frac{0}{0}$)!

Answer: $\boxed{0}$

Verification: The numerator equals zero at $x = 2$, but the denominator does not, so the limit is simply 0.

Note: Not every limit requires complex algebra! Always try direct substitution first.

Definition 1.10: Infinite Limits

We write:

$$\lim_{x\to a} f(x) = \infty$$

if $f(x)$ increases without bound as x approaches a.

We write:

$$\lim_{x\to a} f(x) = -\infty$$

if $f(x)$ decreases without bound (becomes arbitrarily negative) as x approaches a.

Important Notes:

- ∞ is NOT a number; it's a notation describing unbounded behavior
- When we say limit equals ∞, the limit does NOT exist (in the finite sense)
- Vertical asymptotes occur at values where $\lim_{x\to a} f(x) = \pm\infty$

Common Pattern: For $\lim_{x \to a} \frac{f(x)}{g(x)}$:

- If $f(a) \neq 0$ and $g(a) = 0$, then limit is $\pm\infty$ (infinite limit)
- Sign depends on sign of $f(a)$ and whether approaching from left/right

Example 16: Infinite Limits

Problem: Evaluate:

$$\lim_{x \to 3^+} \frac{x+1}{x-3}$$

Solution:
Step 1: Analyze numerator and denominator at $x = 3$
Numerator: $3 + 1 = 4 \neq 0$
Denominator: $3 - 3 = 0$
This suggests an infinite limit.
Step 2: Determine the sign
Since we approach from the right ($x \to 3^+$), we have $x > 3$, so:

- $x + 1 > 0$ (always positive near 3)
- $x - 3 > 0$ (positive when $x > 3$)

Step 3: Create a table approaching from the right

x	$x - 3$	$\frac{x+1}{x-3}$
3.1	0.1	$\frac{4.1}{0.1} = 41$
3.01	0.01	$\frac{4.01}{0.01} = 401$
3.001	0.001	$\frac{4.001}{0.001} = 4001$

Conclusion: As $x \to 3^+$, the fraction grows without bound.
Answer: $\boxed{\lim_{x \to 3^+} \frac{x+1}{x-3} = +\infty}$
⚠ **Note:** From the left ($x \to 3^-$), we would get $-\infty$ because $x - 3 < 0$.

Section 1.3 Summary

Technique	When to Use
Direct Substitution	First attempt; works when function is continuous
Factoring	When you get $\frac{0}{0}$; factor and cancel common terms
Rationalization	Square roots in numerator or denominator; multiply by conjugate
Common Denominator	Complex fractions; combine fractions first
Infinite Limits	Numerator $\neq 0$, denominator $= 0$; limit is $\pm\infty$

Problem-Solving Flowchart:

1. **Try direct substitution**
   - If you get a number $\to$ that's your answer

- If you get $\frac{\text{nonzero}}{0} \rightarrow$ infinite limit
- If you get $\frac{0}{0} \rightarrow$ continue to step 2

2. **Simplify algebraically**
   - Factor if polynomial
   - Rationalize if square roots present
   - Find common denominator if complex fraction
3. **Cancel common factors**
4. **Try direct substitution again**

Key Formulas to Remember:

- Conjugate: $(a - b)(a + b) = a^2 - b^2$
- Difference of squares: $x^2 - a^2 = (x - a)(x + a)$
- Factoring: $x^2 + bx + c = (x + p)(x + q)$ where $p + q = b$, $pq = c$

Indeterminate Forms of Limits

When evaluating limits, we often encounter expressions that don't immediately reveal the answer. These are called indeterminate forms. In this section, we learn to recognize and resolve these special cases.

Definition 1.11: Indeterminate Forms

An expression is called **indeterminate** if its value cannot be determined without further analysis. The most common indeterminate forms are:

Type 1: $\frac{0}{0}$ (zero over zero)
Type 2: $\frac{\infty}{\infty}$ (infinity over infinity)
Type 3: $0 \cdot \infty$ (zero times infinity)
Type 4: $\infty - \infty$ (infinity minus infinity)
Type 5: 0^0, 1^∞, ∞^0 (indeterminate powers)

Why "Indeterminate"?

These forms can have different values depending on the specific functions involved. For example:

- $\lim_{x\to 0} \frac{x}{x} = 1$ (form $\frac{0}{0}$)
- $\lim_{x\to 0} \frac{x^2}{x} = 0$ (form $\frac{0}{0}$)
- $\lim_{x\to 0} \frac{x}{x^2}$ does not exist (form $\frac{0}{0}$)

All three have the same form, but different answers!

Not Indeterminate: $\frac{a}{0}$ where $a \neq 0$ is NOT indeterminate; this gives $\pm\infty$.

Example 17: Resolving $\frac{0}{0}$ by Factoring

Problem: Evaluate:

$$\lim_{h\to 0} \frac{h}{\sin 3h}$$

Solution:

Step 1: Check the form

$$\frac{0}{\sin(0)} = \frac{0}{0}$$

This is indeterminate form $\frac{0}{0}$.

Step 2: Use the standard limit $\lim_{u\to 0} \frac{\sin u}{u} = 1$

Rewrite to match this pattern:

$$\begin{aligned}
\lim_{h\to 0} \frac{h}{\sin 3h} &= \lim_{h\to 0} \frac{h}{\sin 3h} \cdot \frac{3}{3} \\
&= \lim_{h\to 0} \frac{3h}{3\sin 3h} \\
&= \frac{1}{3} \lim_{h\to 0} \frac{3h}{\sin 3h}
\end{aligned}$$

Step 3: Let $u = 3h$, so as $h \to 0$, we have $u \to 0$

$$
\begin{aligned}
&= \frac{1}{3} \lim_{u \to 0} \frac{u}{\sin u} \\
&= \frac{1}{3} \cdot \frac{1}{\lim_{u \to 0} \frac{\sin u}{u}} \\
&= \frac{1}{3} \cdot \frac{1}{1} \\
&= \frac{1}{3}
\end{aligned}
$$

Answer: $\boxed{\frac{1}{3}}$

Key Insight: We used the fact that $\lim_{u \to 0} \frac{u}{\sin u} = \frac{1}{\lim_{u \to 0} \frac{\sin u}{u}} = \frac{1}{1} = 1$

Definition 1.12: Important Trigonometric Limits

These are fundamental limits used frequently in calculus:

Limit 1:

$$\boxed{\lim_{x \to 0} \frac{\sin x}{x} = 1}$$

Limit 2:

$$\boxed{\lim_{x \to 0} \frac{1 - \cos x}{x} = 0}$$

Limit 3:

$$\boxed{\lim_{x \to 0} \frac{\tan x}{x} = 1}$$

Usage:

- These must be memorized!
- x must be in radians, not degrees
- Can be adapted: $\lim_{x \to 0} \frac{\sin(kx)}{kx} = 1$ for any constant $k \neq 0$

Example 18: Trigonometric Limit

Problem: Evaluate:

$$\lim_{x \to 0} \frac{\sin 2x}{5x}$$

Solution:

Step 1: Identify the indeterminate form

$$\frac{\sin 0}{5(0)} = \frac{0}{0}$$

Step 2: Manipulate to use standard limit

We want to create $\frac{\sin 2x}{2x}$:

$$\begin{aligned}\lim_{x\to 0}\frac{\sin 2x}{5x} &= \lim_{x\to 0}\frac{\sin 2x}{5x}\cdot\frac{2}{2}\\ &= \lim_{x\to 0}\frac{2\sin 2x}{10x}\\ &= \frac{2}{10}\lim_{x\to 0}\frac{\sin 2x}{x}\\ &= \frac{1}{5}\lim_{x\to 0}\frac{\sin 2x}{x}\end{aligned}$$

Wait, this isn't quite right. Let me fix it:

$$\begin{aligned}\lim_{x\to 0}\frac{\sin 2x}{5x} &= \lim_{x\to 0}\frac{\sin 2x}{2x}\cdot\frac{2x}{5x}\\ &= \lim_{x\to 0}\frac{\sin 2x}{2x}\cdot\frac{2}{5}\\ &= \left(\lim_{x\to 0}\frac{\sin 2x}{2x}\right)\cdot\frac{2}{5}\end{aligned}$$

Step 3: Apply standard limit
Let $u = 2x$, then as $x \to 0$, $u \to 0$:

$$\lim_{x\to 0}\frac{\sin 2x}{2x} = \lim_{u\to 0}\frac{\sin u}{u} = 1$$

Therefore:

$$= 1\cdot\frac{2}{5} = \frac{2}{5}$$

Answer: $\boxed{\frac{2}{5}}$

Example 19: Hyperbolic Function Limit

Problem: Solve:

$$\frac{5}{4}e^x = \frac{1}{4}\sinh x$$

(This is from 2023 past paper, Q1 part iii)

Solution:

Step 1: Recall definition of $\sinh x$

$$\sinh x = \frac{e^x - e^{-x}}{2}$$

Step 2: Substitute into the equation

$$\begin{aligned}\frac{5}{4}e^x &= \frac{1}{4}\cdot\frac{e^x - e^{-x}}{2}\\ \frac{5}{4}e^x &= \frac{e^x - e^{-x}}{8}\end{aligned}$$

Step 3: Multiply both sides by 8

$$10e^x = e^x - e^{-x}$$

Step 4: Rearrange

$$\begin{aligned} 10e^x - e^x &= -e^{-x} \\ 9e^x &= -e^{-x} \end{aligned}$$

Step 5: Multiply both sides by e^x

$$\begin{aligned} 9e^{2x} &= -1 \\ e^{2x} &= -\frac{1}{9} \end{aligned}$$

Conclusion: Since $e^{2x} > 0$ for all real x, but the right side is negative, there is **no real solution**.

Answer: $\boxed{\text{No solution in } \mathbb{R}}$

⚠ **Note:** This equation has no real solution because the exponential function is always positive.

Definition 1.13: Limits at Infinity

We write:

$$\lim_{x \to \infty} f(x) = L$$

if $f(x)$ approaches L as x increases without bound.

We write:

$$\lim_{x \to -\infty} f(x) = L$$

if $f(x)$ approaches L as x decreases without bound (becomes more negative).

Key Technique for Rational Functions:

For $\lim_{x \to \infty} \frac{P(x)}{Q(x)}$ where P and Q are polynomials:

Step 1: Divide numerator and denominator by highest power of x in denominator

Step 2: Use the fact that $\lim_{x \to \infty} \frac{1}{x^n} = 0$ for $n > 0$

Result depends on degrees:

- If $\deg(P) < \deg(Q)$: limit $= 0$
- If $\deg(P) = \deg(Q)$: limit $=$ ratio of leading coefficients
- If $\deg(P) > \deg(Q)$: limit $= \pm\infty$

Example 20: Limit at Infinity

Problem: Evaluate:

$$\lim_{x \to \infty} \frac{3x^2 - 5x + 2}{2x^2 + x - 1}$$

Solution:

Method 1: Divide by highest power

The highest power in the denominator is x^2. Divide every term by x^2:

$$\lim_{x\to\infty} \frac{3x^2 - 5x + 2}{2x^2 + x - 1} = \lim_{x\to\infty} \frac{\frac{3x^2}{x^2} - \frac{5x}{x^2} + \frac{2}{x^2}}{\frac{2x^2}{x^2} + \frac{x}{x^2} - \frac{1}{x^2}}$$
$$= \lim_{x\to\infty} \frac{3 - \frac{5}{x} + \frac{2}{x^2}}{2 + \frac{1}{x} - \frac{1}{x^2}}$$

Step 2: Apply limit (as $x \to \infty$, terms with x in denominator go to 0)

$$= \frac{3 - 0 + 0}{2 + 0 - 0}$$
$$= \frac{3}{2}$$

Method 2: Quick rule
Since degrees are equal (both degree 2), the limit is the ratio of leading coefficients:

$$\frac{3}{2}$$

Answer: $\boxed{\frac{3}{2}}$

Horizontal Asymptote: The line $y = \frac{3}{2}$ is a horizontal asymptote.

Example 21: Limit at Infinity with Square Root

Problem: Evaluate:

$$\lim_{x\to\infty} \left(\sqrt{x^2 + 4x} - x\right)$$

Solution:
Step 1: Check the form
As $x \to \infty$: $\sqrt{x^2 + 4x} \to \infty$ and $x \to \infty$
So we have the form $\infty - \infty$ (indeterminate).
Step 2: Rationalize by multiplying by conjugate

$$\lim_{x\to\infty} \left(\sqrt{x^2 + 4x} - x\right) \cdot \frac{\sqrt{x^2 + 4x} + x}{\sqrt{x^2 + 4x} + x}$$
$$= \lim_{x\to\infty} \frac{(\sqrt{x^2 + 4x})^2 - x^2}{\sqrt{x^2 + 4x} + x}$$
$$= \lim_{x\to\infty} \frac{x^2 + 4x - x^2}{\sqrt{x^2 + 4x} + x}$$
$$= \lim_{x\to\infty} \frac{4x}{\sqrt{x^2 + 4x} + x}$$

Step 3: Divide numerator and denominator by x
For the square root term: $\sqrt{x^2 + 4x} = \sqrt{x^2(1 + \frac{4}{x})} = x\sqrt{1 + \frac{4}{x}}$ (assuming $x > 0$)

$$= \lim_{x\to\infty} \frac{4x}{x\sqrt{1+\frac{4}{x}}+x}$$

$$= \lim_{x\to\infty} \frac{4x}{x\left(\sqrt{1+\frac{4}{x}}+1\right)}$$

$$= \lim_{x\to\infty} \frac{4}{\sqrt{1+\frac{4}{x}}+1}$$

Step 4: Apply the limit

$$= \frac{4}{\sqrt{1+0}+1}$$

$$= \frac{4}{1+1}$$

$$= \frac{4}{2}$$

$$= 2$$

Answer: $\boxed{2}$

Interpretation: Even though both terms go to infinity, their difference approaches 2.

Example 22: Indeterminate Form Practice

Problem: Evaluate:

$$\lim_{t\to\infty} \left(t^2e^{-t},\ -2\cos(\pi t)\right)$$

(This is a vector limit from 2022 past paper, Q1 part viii)

Solution:

This is a vector with two components. Evaluate each component separately.

Component 1: $\lim_{t\to\infty} t^2e^{-t}$

This has the form $\infty \cdot 0$ (indeterminate).

Rewrite:

$$t^2e^{-t} = \frac{t^2}{e^t}$$

This is form $\frac{\infty}{\infty}$. As $t \to \infty$, e^t grows much faster than t^2, so:

$$\lim_{t\to\infty} \frac{t^2}{e^t} = 0$$

Component 2: $\lim_{t\to\infty} (-2\cos(\pi t))$

As $t \to \infty$, $\cos(\pi t)$ oscillates between -1 and 1, so $-2\cos(\pi t)$ oscillates between -2 and 2.

The limit **does not exist** (oscillation, no convergence).

Answer: The vector limit **does not exist** because the second component does not converge.

However, if this question asks for the limit in a different context (perhaps limit of magnitude), we'd need more information.

⚠ **Note:** Trigonometric functions oscillate and don't have limits as $t \to \infty$ unless they're damped (multiplied by something going to 0).

Section 1.4 Summary

Form	Technique	Example
$\frac{0}{0}$	Factor and cancel, or rationalize	$\lim_{x\to 2} \frac{x^2-4}{x-2}$
$\frac{\infty}{\infty}$	Divide by highest power	$\lim_{x\to\infty} \frac{x^2}{2x^2+1}$
$\infty - \infty$	Rationalize (if square roots)	$\lim_{x\to\infty} (\sqrt{x+1} - x)$
$0 \cdot \infty$	Convert to $\frac{0}{0}$ or $\frac{\infty}{\infty}$	$\lim_{x\to\infty} xe^{-x}$

Essential Memorized Limits:

Limit	Value
$\lim_{x\to 0} \frac{\sin x}{x}$	1
$\lim_{x\to 0} \frac{1-\cos x}{x}$	0
$\lim_{x\to 0} \frac{\tan x}{x}$	1
$\lim_{x\to\infty} \frac{1}{x^n}$	0 (for $n > 0$)

Common Mistakes:

- Assuming $\frac{0}{0} = 0$ (it's indeterminate!)
- Treating $\infty - \infty = 0$ (it's indeterminate!)
- Forgetting to convert degrees to radians for trig limits
- Not recognizing when a limit oscillates (doesn't exist)

Quick Reference for Limits at Infinity:

For $\lim_{x\to\infty} \frac{a_n x^n + \ldots + a_0}{b_m x^m + \ldots + b_0}$:

- If $n < m$: limit $= 0$
- If $n = m$: limit $= \frac{a_n}{b_m}$
- If $n > m$: limit $= \pm\infty$

Special Limits

In this section, we explore important special limits that appear frequently in calculus, particularly those involving trigonometric, exponential, and logarithmic functions. These limits form the foundation for derivatives and integration techniques.

Definition 1.14: Standard Trigonometric Limits

The following limits are fundamental and must be memorized:

Limit 1: Sine Limit

$$\boxed{\lim_{x\to 0} \frac{\sin x}{x} = 1}$$

Limit 2: Cosine Limit

$$\boxed{\lim_{x\to 0} \frac{1-\cos x}{x} = 0}$$

Limit 3: Tangent Limit

$$\boxed{\lim_{x\to 0} \frac{\tan x}{x} = 1}$$

Limit 4: Alternative Cosine Form

$$\boxed{\lim_{x\to 0} \frac{1-\cos x}{x^2} = \frac{1}{2}}$$

Critical Requirements:

- x must be measured in **radians**, not degrees
- These work for any expression approaching 0, not just x
- Example: $\lim_{x\to 0} \frac{\sin(3x)}{3x} = 1$

Warning: If x is in degrees, $\lim_{x\to 0} \frac{\sin x}{x} = \frac{\pi}{180} \neq 1$

Example 23: Trigonometric Limit with Coefficient

Problem: Evaluate:

$$\lim_{x\to 0} \frac{\sin 2x}{5x}$$

(From 2021 past paper, Q1 part v)

Solution:

Step 1: Identify the form

$$\frac{\sin 0}{5(0)} = \frac{0}{0} \quad \text{(indeterminate)}$$

Step 2: Rewrite to match standard form $\frac{\sin u}{u}$

We need the argument of sine to match the denominator. Multiply and divide by 2:

$$\begin{aligned}\lim_{x\to 0}\frac{\sin 2x}{5x} &= \lim_{x\to 0}\frac{\sin 2x}{5x}\cdot\frac{2}{2}\\ &= \lim_{x\to 0}\frac{2\sin 2x}{10x}\\ &= \frac{2}{10}\lim_{x\to 0}\frac{\sin 2x}{x}\end{aligned}$$

Wait, that's not quite right. Let me recalculate:

$$\begin{aligned}\lim_{x\to 0}\frac{\sin 2x}{5x} &= \frac{1}{5}\lim_{x\to 0}\frac{\sin 2x}{x}\\ &= \frac{1}{5}\lim_{x\to 0}\frac{\sin 2x}{2x}\cdot 2\\ &= \frac{1}{5}\cdot 2\cdot\lim_{x\to 0}\frac{\sin 2x}{2x}\end{aligned}$$

Step 3: Apply standard limit
Let $u = 2x$. As $x \to 0$, we have $u \to 0$:

$$\lim_{x\to 0}\frac{\sin 2x}{2x} = \lim_{u\to 0}\frac{\sin u}{u} = 1$$

Step 4: Complete the calculation

$$= \frac{1}{5}\cdot 2\cdot 1 = \frac{2}{5}$$

Answer: $\boxed{\frac{2}{5}}$

Key Strategy: Always manipulate to get $\dfrac{\sin(\textit{something})}{\textit{same something}}$

Example 24: Multiple Trigonometric Terms

Problem: Evaluate:

$$\lim_{h\to 0}\frac{h}{\sin 3h}$$

Solution:
Step 1: Check the form

$$\frac{0}{\sin 0} = \frac{0}{0}$$

Step 2: Rewrite as reciprocal of standard form
We want to use $\lim\limits_{u\to 0}\dfrac{\sin u}{u} = 1$, so:

$$\begin{aligned}\lim_{h\to 0}\frac{h}{\sin 3h} &= \lim_{h\to 0}\frac{1}{\frac{\sin 3h}{h}}\\ &= \frac{1}{\lim\limits_{h\to 0}\frac{\sin 3h}{h}}\end{aligned}$$

Step 3: Manipulate denominator limit

$$\begin{aligned}\lim_{h\to 0}\frac{\sin 3h}{h} &= \lim_{h\to 0}\frac{\sin 3h}{3h}\cdot 3\\ &= 3\cdot\lim_{h\to 0}\frac{\sin 3h}{3h}\\ &= 3\cdot 1\\ &= 3\end{aligned}$$

Step 4: Calculate final answer

$$\lim_{h\to 0}\frac{h}{\sin 3h} = \frac{1}{3}$$

Answer: $\boxed{\frac{1}{3}}$

Verification: Check with small value: $h = 0.01$, $\frac{0.01}{\sin(0.03)} = \frac{0.01}{0.02999...} \approx 0.333$ ✓

Definition 1.15: The Natural Exponential Limit

The most important limit in exponential functions is:

Euler's Limit (Form 1):

$$\boxed{\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e}$$

where $e \approx 2.71828...$ is Euler's number.

Euler's Limit (Form 2):

$$\boxed{\lim_{x\to 0}(1+x)^{1/x} = e}$$

General Form:

$$\boxed{\lim_{x\to 0}\frac{e^x - 1}{x} = 1}$$

Logarithmic Form:

$$\boxed{\lim_{x\to 0}\frac{\ln(1+x)}{x} = 1}$$

Applications:

- These limits define the derivatives of e^x and $\ln x$
- Used extensively in compound interest and continuous growth models
- Foundation for L'Hôpital's Rule (advanced calculus)

Example 25: Proving Euler's Limit

Problem: Show that:

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e$$

(From 2024 past paper, Q6 part b)

Solution:

This is one of the most important limits in calculus. We'll prove it using the definition of e.

Step 1: Take natural logarithm of both sides

Let $L = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$. We want to show $L = e$.

Taking logarithm:

$$\begin{aligned}\ln L &= \ln\left[\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n\right] \\ &= \lim_{n\to\infty}\ln\left[\left(1+\frac{1}{n}\right)^n\right] \\ &= \lim_{n\to\infty} n\ln\left(1+\frac{1}{n}\right)\end{aligned}$$

Step 2: Rewrite as a fraction

This has the form $\infty \cdot 0$. Convert to $\frac{0}{0}$:

$$\ln L = \lim_{n\to\infty}\frac{\ln\left(1+\frac{1}{n}\right)}{\frac{1}{n}}$$

Step 3: Use substitution

Let $u = \frac{1}{n}$. As $n \to \infty$, $u \to 0$:

$$\ln L = \lim_{u\to 0}\frac{\ln(1+u)}{u}$$

Step 4: Apply standard logarithmic limit

By definition (or using L'Hôpital's Rule, which we'll learn later):

$$\lim_{u\to 0}\frac{\ln(1+u)}{u} = 1$$

Therefore: $\ln L = 1$

Step 5: Solve for L

$$\begin{aligned}\ln L &= 1 \\ L &= e^1 \\ L &= e\end{aligned}$$

Conclusion: $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e$ ✓

Answer: Proven. $\boxed{Q.E.D.}$

Example 26: Exponential Limit

Problem: Evaluate:

$$\lim_{x \to 0} \frac{e^{5x} - 1}{x}$$

Solution:

Step 1: Identify the form

$$\frac{e^0 - 1}{0} = \frac{0}{0}$$

Step 2: Use the standard exponential limit

We know $\lim_{u \to 0} \frac{e^u - 1}{u} = 1$

Rewrite our limit to match this form:

$$\lim_{x \to 0} \frac{e^{5x} - 1}{x} = \lim_{x \to 0} \frac{e^{5x} - 1}{5x} \cdot 5$$
$$= 5 \cdot \lim_{x \to 0} \frac{e^{5x} - 1}{5x}$$

Step 3: Substitute $u = 5x$

As $x \to 0$, $u \to 0$:

$$= 5 \cdot \lim_{u \to 0} \frac{e^u - 1}{u}$$
$$= 5 \cdot 1$$
$$= 5$$

Answer: $\boxed{5}$

Pattern: $\lim_{x \to 0} \frac{e^{ax} - 1}{x} = a$

Definition 1.16: Logarithmic Limits

Important limits involving logarithmic functions:

Limit 1: Basic Logarithm

$$\boxed{\lim_{x \to 0} \frac{\ln(1 + x)}{x} = 1}$$

Limit 2: General Base

$$\boxed{\lim_{x \to 0} \frac{a^x - 1}{x} = \ln a}$$

where $a > 0$, $a \neq 1$.

Limit 3: Power of Logarithm

$$\boxed{\lim_{x \to 0^+} x \ln x = 0}$$

Important Notes:

- ln always means natural logarithm (base e)
- For base 10: $\log_{10} x = \frac{\ln x}{\ln 10}$
- The limit $\lim_{x \to 0^+} x \ln x$ has form $0 \cdot (-\infty)$ but equals 0

Example 27: Logarithmic Limit with Base a

Problem: Show that:

$$\lim_{x\to 0}\frac{a^x-1}{x}=\log_e a=\ln a$$

(From 2023 past paper, Q6 part b)

Solution:

Step 1: Rewrite a^x in terms of e

Recall that $a=e^{\ln a}$, so:

$$a^x=(e^{\ln a})^x=e^{x\ln a}$$

Step 2: Substitute into the limit

$$\lim_{x\to 0}\frac{a^x-1}{x}=\lim_{x\to 0}\frac{e^{x\ln a}-1}{x}$$

Step 3: Use substitution

Let $u=x\ln a$. As $x\to 0$, $u\to 0$. Also, $x=\frac{u}{\ln a}$:

$$\begin{aligned}&=\lim_{u\to 0}\frac{e^u-1}{\frac{u}{\ln a}}\\&=\lim_{u\to 0}\frac{e^u-1}{u}\cdot\ln a\\&=\ln a\cdot\lim_{u\to 0}\frac{e^u-1}{u}\end{aligned}$$

Step 4: Apply standard exponential limit

$$=\ln a\cdot 1=\ln a$$

Conclusion: $\lim_{x\to 0}\frac{a^x-1}{x}=\ln a$ ✓

Answer: Proven. $\boxed{Q.E.D.}$

Application: This proves that $\frac{d}{dx}(a^x)=a^x\ln a$

Example 28: Combined Trigonometric and Algebraic Limit

Problem: Evaluate:

$$\lim_{\theta\to 0}\frac{1-\cos\theta}{\theta^2}$$

Solution:

Method 1: Using Trigonometric Identity

Use the identity $1-\cos\theta=2\sin^2\left(\frac{\theta}{2}\right)$:

$$\lim_{\theta\to 0}\frac{1-\cos\theta}{\theta^2}=\lim_{\theta\to 0}\frac{2\sin^2\left(\frac{\theta}{2}\right)}{\theta^2}$$

Rewrite:

$$\begin{aligned}
&= \lim_{\theta\to 0} \frac{2\sin^2\left(\frac{\theta}{2}\right)}{\theta^2} \\
&= 2\lim_{\theta\to 0} \frac{\sin^2\left(\frac{\theta}{2}\right)}{\theta^2} \\
&= 2\lim_{\theta\to 0} \left[\frac{\sin\left(\frac{\theta}{2}\right)}{\theta}\right]^2 \\
&= 2\lim_{\theta\to 0} \left[\frac{\sin\left(\frac{\theta}{2}\right)}{\frac{\theta}{2}} \cdot \frac{1}{2}\right]^2 \\
&= 2\lim_{\theta\to 0} \left[\frac{1}{2} \cdot \frac{\sin\left(\frac{\theta}{2}\right)}{\frac{\theta}{2}}\right]^2
\end{aligned}$$

Let $u = \frac{\theta}{2}$. As $\theta \to 0$, $u \to 0$:

$$\begin{aligned}
&= 2\left[\frac{1}{2} \cdot \lim_{u\to 0} \frac{\sin u}{u}\right]^2 \\
&= 2\left[\frac{1}{2} \cdot 1\right]^2 \\
&= 2 \cdot \frac{1}{4} \\
&= \frac{1}{2}
\end{aligned}$$

Answer: $\boxed{\frac{1}{2}}$

Quick Fact: This is a standard limit: $\lim_{x\to 0} \frac{1-\cos x}{x^2} = \frac{1}{2}$ (memorize it!)

Example 29: Squeeze Theorem Application

Problem: Evaluate:

$$\lim_{x\to 0} x^2 \sin\left(\frac{1}{x}\right)$$

Solution:

Step 1: Analyze the behavior
As $x \to 0$, $\frac{1}{x} \to \pm\infty$, so $\sin\left(\frac{1}{x}\right)$ oscillates wildly between -1 and 1.
Direct evaluation doesn't work. Use the Squeeze Theorem.
Step 2: Apply bounds for sine
We know: $-1 \le \sin\left(\frac{1}{x}\right) \le 1$ for all $x \ne 0$.
Multiply all parts by x^2 (note: $x^2 \ge 0$):

$$-x^2 \le x^2 \sin\left(\frac{1}{x}\right) \le x^2$$

Step 3: Find limits of bounds

$$\lim_{x\to 0}(-x^2) = 0$$
$$\lim_{x\to 0} x^2 = 0$$

Step 4: Apply Squeeze Theorem
Since the lower bound and upper bound both approach 0:

$$\lim_{x\to 0} x^2 \sin\left(\frac{1}{x}\right) = 0$$

Answer: $\boxed{0}$
Key Idea: Even though $\sin\left(\frac{1}{x}\right)$ oscillates, it's squeezed to 0 by the x^2 term.

Section 1.5 Summary: Essential Limits to Memorize

Limit	Value
$\lim_{x\to 0} \frac{\sin x}{x}$	1
$\lim_{x\to 0} \frac{\tan x}{x}$	1
$\lim_{x\to 0} \frac{1-\cos x}{x}$	0
$\lim_{x\to 0} \frac{1-\cos x}{x^2}$	$\frac{1}{2}$
$\lim_{x\to 0} \frac{e^x - 1}{x}$	1
$\lim_{x\to 0} \frac{\ln(1+x)}{x}$	1
$\lim_{x\to 0} \frac{a^x - 1}{x}$	$\ln a$
$\lim_{n\to \infty} \left(1+\frac{1}{n}\right)^n$	e

Useful Generalizations:

- $\lim_{x\to 0} \frac{\sin(ax)}{x} = a$
- $\lim_{x\to 0} \frac{\sin(ax)}{bx} = \frac{a}{b}$
- $\lim_{x\to 0} \frac{e^{ax} - 1}{x} = a$
- $\lim_{x\to 0} \frac{\ln(1+ax)}{x} = a$

Common Errors:

- Using degrees instead of radians in trig limits
- Confusing $\ln a$ with $\log_{10} a$
- Forgetting that $e \approx 2.718$, not 2 or 3
- Not recognizing when to apply Squeeze Theorem

Continuity and Discontinuity

Continuity is a fundamental property of functions that ensures there are no sudden jumps, breaks, or holes in the graph. Understanding continuity is essential for applying calculus theorems and solving real-world problems.

Definition 1.17: Continuity at a Point

A function f is **continuous at a point** $x = a$ if three conditions are satisfied:

Condition 1: $f(a)$ exists (the function is defined at a)
Condition 2: $\lim\limits_{x \to a} f(x)$ exists (the limit exists at a)
Condition 3: $\lim\limits_{x \to a} f(x) = f(a)$ (the limit equals the function value)

In Simple Terms:

A function is continuous at a if you can draw its graph through $x = a$ without lifting your pen.

Equivalent Statement:

f is continuous at a if and only if:

$$\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x) = f(a)$$

All three values (left limit, right limit, and function value) must be equal.

Definition 1.18: Types of Discontinuity

If f is not continuous at $x = a$, we say f has a **discontinuity** at a. There are three main types:

Type 1: Removable Discontinuity

- $\lim\limits_{x \to a} f(x)$ exists, but either $f(a)$ is undefined or $f(a) \neq \lim_{x \to a} f(x)$
- Can be "fixed" by redefining $f(a)$
- Example: $f(x) = \frac{x^2-4}{x-2}$ at $x = 2$

Type 2: Jump Discontinuity

- Left and right limits exist but are not equal: $\lim\limits_{x \to a^-} f(x) \neq \lim\limits_{x \to a^+} f(x)$
- Graph has a "jump" at $x = a$
- Example: Step functions, piecewise functions

Type 3: Infinite Discontinuity

- At least one of the one-sided limits is infinite
- Vertical asymptote at $x = a$
- Example: $f(x) = \frac{1}{x-2}$ at $x = 2$

Example 30: Testing Continuity

Problem: Determine whether $f(x) = \frac{x^2-9}{x-3}$ is continuous at $x = 3$. If not, what type of discontinuity exists?

Solution:
Step 1: Check if $f(3)$ exists

$$f(3) = \frac{3^2 - 9}{3 - 3} = \frac{0}{0} \quad \text{(undefined)}$$

Condition 1 fails! ×
Already we know the function is NOT continuous at $x = 3$.
Step 2: Find $\lim_{x\to 3} f(x)$
Factor:

$$\begin{aligned} f(x) &= \frac{x^2 - 9}{x - 3} \\ &= \frac{(x-3)(x+3)}{x-3} \\ &= x + 3 \quad (\text{for } x \neq 3) \end{aligned}$$

Therefore:

$$\lim_{x\to 3} f(x) = \lim_{x\to 3}(x+3) = 6$$

Step 3: Classify the discontinuity
Since $\lim_{x\to 3} f(x) = 6$ exists (finite), but $f(3)$ is undefined, this is a **removable discontinuity**.
Answer: The function is **NOT continuous** at $x = 3$. It has a **removable discontinuity** (a hole in the graph at the point $(3, 6)$).
How to Fix It: Define $g(x)$ as:

$$g(x) = \begin{cases} \frac{x^2-9}{x-3}, & x \neq 3 \\ 6, & x = 3 \end{cases}$$

Then $g(x)$ is continuous everywhere! ✓

Example 31: Continuous Extension

Problem: Define $g(x)$ in a way that extends $g(x) = \frac{x^2-9}{x-3}$ to be continuous at $x = 3$.
(From 2023 past paper, Q1 part vi and 2024 past paper, Q1 part xii)

Solution:
Step 1: Find the limit at $x = 3$
From the previous example, we found:

$$\lim_{x\to 3} \frac{x^2 - 9}{x - 3} = 6$$

Step 2: Define the continuous extension
For $g(x)$ to be continuous at $x = 3$, we need:

$$g(3) = \lim_{x\to 3} g(x) = 6$$

Answer: Define:

$$\boxed{g(x) = \begin{cases} \frac{x^2-9}{x-3}, & x \neq 3 \\ 6, & x = 3 \end{cases}}$$

Or equivalently: $\boxed{g(x) = x + 3}$ for all x (since this equals $\frac{x^2-9}{x-3}$ for $x \neq 3$).

Verification:

- $g(3) = 6$ ✓
- $\lim_{x\to 3} g(x) = 6$ ✓
- $g(3) = \lim_{x\to 3} g(x)$ ✓

All three conditions for continuity are satisfied!

Example 32: Piecewise Function Continuity

Problem: For what value of b is the function

$$f(x) = \begin{cases} x, & x < -2 \\ bx^2, & x \geq -2 \end{cases}$$

continuous at every x?

(From 2023 past paper, Q2 part b)

Solution:

The function is automatically continuous everywhere except possibly at $x = -2$ (the transition point). We need to ensure continuity there.

Step 1: Find left-hand limit

As $x \to -2$ from the left, use $f(x) = x$:

$$\lim_{x\to -2^-} f(x) = \lim_{x\to -2^-} x = -2$$

Step 2: Find right-hand limit

As $x \to -2$ from the right, use $f(x) = bx^2$:

$$\lim_{x\to -2^+} f(x) = \lim_{x\to -2^+} bx^2 = b(-2)^2 = 4b$$

Step 3: Find function value at $x = -2$

Since $-2 \geq -2$, we use the second piece:

$$f(-2) = b(-2)^2 = 4b$$

Step 4: Set up continuity condition

For continuity at $x = -2$:

$$\lim_{x\to -2^-} f(x) = \lim_{x\to -2^+} f(x) = f(-2)$$

This gives:

$$-2 = 4b = 4b$$

The equation $-2 = 4b$ gives:

$$b = -\frac{2}{4} = -\frac{1}{2}$$

Answer: $\boxed{b = -\frac{1}{2}}$

Verification: With $b = -\frac{1}{2}$:

- Left limit: -2
- Right limit: $4(-\frac{1}{2}) = -2$
- Function value: $f(-2) = -\frac{1}{2}(4) = -2$

All equal! ✓

Example 33: Finding Continuity Value (Another Type)

Problem: For what value of a is

$$f(x) = \begin{cases} x^2 - 1, & x < 3 \\ 2ax, & x \geq 3 \end{cases}$$

continuous at every x?

(From 2021 past paper, Q2 part b)

Solution:

Step 1: Find left-hand limit at $x = 3$

$$\lim_{x \to 3^-} f(x) = \lim_{x \to 3^-} (x^2 - 1) = 3^2 - 1 = 9 - 1 = 8$$

Step 2: Find right-hand limit at $x = 3$

$$\lim_{x \to 3^+} f(x) = \lim_{x \to 3^+} 2ax = 2a(3) = 6a$$

Step 3: Find function value at $x = 3$

$$f(3) = 2a(3) = 6a \quad (\text{since } 3 \geq 3)$$

Step 4: Apply continuity condition

$$\lim_{x \to 3^-} f(x) = \lim_{x \to 3^+} f(x)$$
$$8 = 6a$$
$$a = \frac{8}{6} = \frac{4}{3}$$

Answer: $\boxed{a = \frac{4}{3}}$

Check: With $a = \frac{4}{3}$:

$$\text{Left limit: } 8$$
$$\text{Right limit: } 6 \cdot \frac{4}{3} = 8$$
$$\text{Function value: } f(3) = 2 \cdot \frac{4}{3} \cdot 3 = 8$$

All equal! ✓

Definition 1.19: Continuity on an Interval

A function f is **continuous on an interval** if it is continuous at every point in that interval.

Continuous on (a, b)**:** Continuous at every point x where $a < x < b$

Continuous on $[a, b]$**:** Continuous at every point in (a, b), and:

- $\lim_{x \to a^+} f(x) = f(a)$ (continuous from the right at a)
- $\lim_{x \to b^-} f(x) = f(b)$ (continuous from the left at b)

Important Continuous Functions:

The following types of functions are continuous on their domains:

- Polynomials: continuous everywhere ($\mathbb{R}$)
- Rational functions: continuous wherever denominator $\neq 0$
- Trigonometric functions: $\sin x$, $\cos x$ continuous on $\mathbb{R}$; $\tan x$ discontinuous at $x = \frac{\pi}{2} + n\pi$
- Exponential functions: e^x, a^x continuous on $\mathbb{R}$
- Logarithmic functions: $\ln x$ continuous on $(0, \infty)$

Example 34: Determining Continuity Domain

Problem: At what points is the function $f(x) = \frac{1}{x^2 - 3x}$ continuous?
(From 2023 past paper, Q1 part vii)

Solution:

Step 1: Identify where the function is defined
The function is undefined where the denominator equals zero:

$$\begin{aligned} x^2 - 3x &= 0 \\ x(x-3) &= 0 \\ x = 0 \quad &\text{or} \quad x = 3 \end{aligned}$$

Step 2: Determine continuity
Since f is a rational function, it is continuous wherever it is defined.
The function is defined for all x except $x = 0$ and $x = 3$.
Answer: The function is continuous on:

$$\boxed{(-\infty, 0) \cup (0, 3) \cup (3, \infty)}$$

Or equivalently: $\boxed{\mathbb{R} \setminus \{0, 3\}}$

Note: At $x = 0$ and $x = 3$, the function has **infinite discontinuities** (vertical asymptotes).

Theorem 1.1: Properties of Continuous Functions

If f and g are continuous at $x = a$, then the following are also continuous at $x = a$:

1. **Sum:** $f + g$
2. **Difference:** $f - g$
3. **Constant Multiple:** cf for any constant c
4. **Product:** $f \cdot g$
5. **Quotient:** $\dfrac{f}{g}$, provided $g(a) \neq 0$
6. **Composition:** $f \circ g$, provided g is continuous at a and f is continuous at $g(a)$

Example 35: Proving Continuity Using Properties

Problem: If f and g are continuous at a, prove that fg is also continuous at a.
(From 2022 past paper, Q1 part iii)

Solution:
To prove fg is continuous at a, we must show:

$$\lim_{x\to a}[f(x) \cdot g(x)] = f(a) \cdot g(a)$$

Given Information:

- f is continuous at a, so $\lim_{x\to a} f(x) = f(a)$
- g is continuous at a, so $\lim_{x\to a} g(x) = g(a)$

Proof:
Using the Product Rule for Limits:

$$\begin{aligned}\lim_{x\to a}[f(x) \cdot g(x)] &= \left[\lim_{x\to a} f(x)\right] \cdot \left[\lim_{x\to a} g(x)\right] \\ &= f(a) \cdot g(a)\end{aligned}$$

Since $\lim_{x\to a}[f(x) \cdot g(x)] = f(a) \cdot g(a)$, the product fg is continuous at a.
Conclusion: fg is continuous at a. ✓
Answer: Proven. $\boxed{Q.E.D.}$

Section 1.6 Summary

Concept	Definition/Property
Continuity at a	$\lim_{x\to a} f(x) = f(a)$ (three conditions must hold)
Removable Discontinuity	Limit exists but $\neq f(a)$ or $f(a)$ undefined; can be fixed
Jump Discontinuity	Left and right limits exist but are unequal
Infinite Discontinuity	At least one one-sided limit is $\pm\infty$
Continuous Extension	Redefine function at a point to make it continuous

Testing Continuity - Checklist:
For piecewise function at transition point $x = a$:

1. Find $f(a)$ (function value)

2. Find $\lim\limits_{x \to a^-} f(x)$ (left limit)
3. Find $\lim\limits_{x \to a^+} f(x)$ (right limit)
4. Check if all three are equal

If equal $\rightarrow$ continuous ✓
If not equal $\rightarrow$ discontinuous ×

Common Exam Questions:

- Find constant to make piecewise function continuous
- Identify type of discontinuity
- Define continuous extension at a removable discontinuity
- State domain of continuity for rational functions

Functions Always Continuous on Their Domains:

- Polynomials
- Rational functions
- $\sin x$, $\cos x$, e^x
- $\ln x$ (on $(0, \infty)$)
- Compositions of continuous functions

Epsilon-Delta Definition (Precise Limits)

While we have worked with limits intuitively, a rigorous mathematical treatment requires a precise definition. The epsilon-delta definition, developed by mathematicians like Cauchy and Weierstrass, provides this precision and is fundamental to advanced calculus.

Definition 1.20: Precise Definition of a Limit (Epsilon-Delta)

Let f be a function defined on an open interval containing a (except possibly at a itself). We say:

$$\lim_{x \to a} f(x) = L$$

if for every $\varepsilon > 0$ (no matter how small), there exists a $\delta > 0$ such that:

$$\text{if } 0 < |x - a| < \delta, \text{ then } |f(x) - L| < \varepsilon$$

In Words:

We can make $f(x)$ arbitrarily close to L (within ε) by taking x sufficiently close to a (within δ), but not equal to a.

Understanding the Components:

- ε (epsilon): Represents how close we want $f(x)$ to be to L (precision requirement)
- δ (delta): Represents how close x must be to a to achieve that precision
- $|x - a|$: Distance from x to a (in the domain)
- $|f(x) - L|$: Distance from $f(x)$ to L (in the range)
- $0 < |x - a|$: Ensures $x \neq a$ (we approach but don't reach a)

Key Insight:

For ANY challenge $\varepsilon > 0$ (no matter how tiny), we must find a response $\delta > 0$ that works. This is a "for all ε, there exists δ" statement.

Definition 1.21: Epsilon-Delta for Limit at Infinity

We say:

$$\lim_{x \to \infty} f(x) = L$$

if for every $\varepsilon > 0$, there exists a number N such that:

$$\text{if } x > N, \text{ then } |f(x) - L| < \varepsilon$$

Similarly, for infinite limits:

We say:

$$\lim_{x \to a} f(x) = \infty$$

if for every $M > 0$, there exists a $\delta > 0$ such that:

$$\text{if } 0 < |x - a| < \delta, \text{ then } f(x) > M$$

Interpretation:

- For $x \to \infty$: We need x large enough (beyond N)
- For limit $= \infty$: We need $f(x)$ arbitrarily large (exceeding any M)

Example 36: Proving a Simple Limit Using Epsilon-Delta

Problem: Use the precise definition to prove that:

$$\lim_{x\to 3}(2x+1) = 7$$

Solution:
We need to show: For every $\varepsilon > 0$, there exists $\delta > 0$ such that:

$$\text{if } 0 < |x-3| < \delta, \text{ then } |(2x+1)-7| < \varepsilon$$

Step 1: Preliminary Analysis
Work backwards from what we want:

$$\begin{aligned} |(2x+1)-7| &< \varepsilon \\ |2x-6| &< \varepsilon \\ |2(x-3)| &< \varepsilon \\ 2|x-3| &< \varepsilon \\ |x-3| &< \frac{\varepsilon}{2} \end{aligned}$$

This suggests we should choose $\delta = \frac{\varepsilon}{2}$.
Step 2: Formal Proof
Let $\varepsilon > 0$ be given (arbitrary). Choose $\delta = \frac{\varepsilon}{2}$.
Now suppose $0 < |x-3| < \delta$. Then:

$$\begin{aligned} |(2x+1)-7| &= |2x-6| \\ &= |2(x-3)| \\ &= 2|x-3| \\ &< 2\delta \quad (\text{since } |x-3| < \delta) \\ &= 2\cdot\frac{\varepsilon}{2} \\ &= \varepsilon \end{aligned}$$

Therefore, $|(2x+1)-7| < \varepsilon$.
Conclusion: We have shown that for every $\varepsilon > 0$, choosing $\delta = \frac{\varepsilon}{2}$ works. Thus, by the epsilon-delta definition:

$$\lim_{x\to 3}(2x+1) = 7 \quad \checkmark$$

Answer: Proven. $\boxed{Q.E.D.}$

Strategy: Work backwards (scratch work) to find δ, then write the formal proof forwards.

Example 37: Epsilon-Delta with Quadratic Function

Problem: Prove using the epsilon-delta definition that:

$$\lim_{x\to 2} x^2 = 4$$

Solution:

We need: For every $\varepsilon > 0$, there exists $\delta > 0$ such that:

$$\text{if } 0 < |x-2| < \delta, \text{ then } |x^2-4| < \varepsilon$$

Step 1: Preliminary Analysis (Scratch Work)

$$\begin{aligned} |x^2-4| &< \varepsilon \\ |x-2||x+2| &< \varepsilon \end{aligned}$$

We can control $|x-2|$ with δ, but what about $|x+2|$?
Restrict x to be close to 2. Say $|x-2| < 1$, then:

$$\begin{aligned} -1 < x-2 &< 1 \\ 1 < x &< 3 \\ 3 < x+2 &< 5 \end{aligned}$$

So $|x+2| < 5$ when $|x-2| < 1$.
Therefore:

$$|x^2-4| = |x-2||x+2| < 5|x-2|$$

For this to be less than ε, we need:

$$5|x-2| < \varepsilon \implies |x-2| < \frac{\varepsilon}{5}$$

So we choose $\delta = \min\left\{1, \frac{\varepsilon}{5}\right\}$.
Step 2: Formal Proof
Let $\varepsilon > 0$ be given. Choose $\delta = \min\left\{1, \frac{\varepsilon}{5}\right\}$.
Suppose $0 < |x-2| < \delta$. Then:
Since $\delta \le 1$, we have $|x-2| < 1$, which implies $|x+2| < 5$ (as shown above).
Since $\delta \le \frac{\varepsilon}{5}$, we have $|x-2| < \frac{\varepsilon}{5}$.
Therefore:

$$\begin{aligned} |x^2-4| &= |(x-2)(x+2)| \\ &= |x-2| \cdot |x+2| \\ &< \frac{\varepsilon}{5} \cdot 5 \\ &= \varepsilon \end{aligned}$$

Conclusion: By the epsilon-delta definition:

$$\lim_{x\to 2} x^2 = 4 \quad \checkmark$$

Answer: Proven. $\boxed{Q.E.D.}$

⚠ **Key Technique:** For polynomial limits, restrict $\delta \le 1$ first to bound variable terms.

Example 38: Proving Infinite Limit

Problem: Use the precise definition to prove that:

$$\lim_{x \to \infty} (x+3)^3 = \infty$$

(From 2024 past paper, Q6 part b)

Solution:
We need to show: For every $M > 0$, there exists N such that:

$$\text{if } x > N, \text{ then } (x+3)^3 > M$$

Step 1: Preliminary Analysis
We want $(x+3)^3 > M$.
Taking cube root:

$$x + 3 > \sqrt[3]{M}$$
$$x > \sqrt[3]{M} - 3$$

This suggests choosing $N = \sqrt[3]{M} - 3$.
Step 2: Formal Proof
Let $M > 0$ be given (arbitrary). Choose $N = \sqrt[3]{M} - 3$.
Now suppose $x > N = \sqrt[3]{M} - 3$. Then:

$$\begin{aligned} x &> \sqrt[3]{M} - 3 \\ x + 3 &> \sqrt[3]{M} \\ (x+3)^3 &> (\sqrt[3]{M})^3 \quad \text{(cubing preserves inequality for positive numbers)} \\ (x+3)^3 &> M \end{aligned}$$

Therefore, $(x+3)^3 > M$.
Conclusion: For every $M > 0$, we found $N = \sqrt[3]{M} - 3$ such that $x > N$ implies $(x+3)^3 > M$. Thus:

$$\lim_{x \to \infty} (x+3)^3 = \infty \quad \checkmark$$

Answer: Proven. $\boxed{Q.E.D.}$

Theorem 1.2: Uniqueness of Limits

If $\lim_{x \to a} f(x) = L$ and $\lim_{x \to a} f(x) = M$, then $L = M$.
In words: A function cannot have two different limits at the same point. The limit, if it exists, is unique.

Proof of Uniqueness of Limits

Proof:
Suppose $\lim_{x \to a} f(x) = L$ and $\lim_{x \to a} f(x) = M$. We will prove $L = M$ by contradiction.
Assume: $L \neq M$ (for contradiction).
Let $\varepsilon = \frac{|L-M|}{2} > 0$ (since $L \neq M$).

Step 1: Since $\lim_{x \to a} f(x) = L$, there exists $\delta_1 > 0$ such that:

$$\text{if } 0 < |x - a| < \delta_1, \text{ then } |f(x) - L| < \varepsilon$$

Step 2: Since $\lim_{x \to a} f(x) = M$, there exists $\delta_2 > 0$ such that:

$$\text{if } 0 < |x - a| < \delta_2, \text{ then } |f(x) - M| < \varepsilon$$

Step 3: Choose $\delta = \min\{\delta_1, \delta_2\}$. Pick any x with $0 < |x - a| < \delta$.
Then both conditions hold:

$$|f(x) - L| < \varepsilon$$
$$|f(x) - M| < \varepsilon$$

Step 4: By the triangle inequality:

$$\begin{aligned}
|L - M| &= |L - f(x) + f(x) - M| \\
&\leq |L - f(x)| + |f(x) - M| \\
&= |f(x) - L| + |f(x) - M| \\
&< \varepsilon + \varepsilon \\
&= 2\varepsilon \\
&= 2 \cdot \frac{|L - M|}{2} \\
&= |L - M|
\end{aligned}$$

This gives $|L - M| < |L - M|$, which is a **contradiction**!
Conclusion: Our assumption that $L \neq M$ must be false. Therefore, $L = M$. □

Example 39: Proving a Limit Does Not Exist

Problem: Show that $\lim_{x \to 0} \frac{1}{x}$ does not exist.

Solution:
We show the limit doesn't exist by demonstrating that one-sided limits are different (one is $+\infty$, the other is $-\infty$).
Approach 1: Using One-Sided Limits
Right-hand limit: As $x \to 0^+$, $x > 0$, so $\frac{1}{x} \to +\infty$.
For any $M > 0$, choose $\delta = \frac{1}{M}$. If $0 < x < \delta$, then:

$$\frac{1}{x} > \frac{1}{\delta} = M$$

So $\lim_{x \to 0^+} \frac{1}{x} = +\infty$.
Left-hand limit: As $x \to 0^-$, $x < 0$, so $\frac{1}{x} \to -\infty$.
For any $M > 0$, choose $\delta = \frac{1}{M}$. If $-\delta < x < 0$, then:

$$\frac{1}{x} < -\frac{1}{\delta} = -M$$

So $\lim_{x \to 0^-} \frac{1}{x} = -\infty$.

Conclusion: Since the one-sided limits are different ($+\infty \neq -\infty$), the two-sided limit $\lim_{x\to 0} \frac{1}{x}$ does not exist.

Answer: $\boxed{\text{Limit does not exist}}$

Note: Even though we say "the limit is $\pm\infty$," technically this means the limit doesn't exist in the finite sense.

Section 1.7 Summary

Epsilon-Delta Definition:

$$\boxed{\lim_{x\to a} f(x) = L \iff \forall\varepsilon > 0, \exists\delta > 0 : 0 < |x-a| < \delta \implies |f(x) - L| < \varepsilon}$$

Proof Strategy (Epsilon-Delta):

1. **Scratch work:** Start with $|f(x) - L| < \varepsilon$ and work backwards to find what δ should be
2. **Restriction:** For polynomials/complicated functions, first restrict $\delta \leq 1$ to bound terms
3. **Formal proof:** Write forwards:
   - "Let $\varepsilon > 0$ be given"
   - "Choose $\delta = ...$"
   - "Suppose $0 < |x-a| < \delta$"
   - Show $|f(x) - L| < \varepsilon$

Common Delta Choices:

Function Type	Typical δ
Linear: $mx + b$	$\delta = \frac{\varepsilon}{\lvert m\rvert}$
Quadratic: x^2	$\delta = \min\left\{1, \frac{\varepsilon}{2a+1}\right\}$
Rational: $\frac{p(x)}{q(x)}$	Restrict first, then find δ

Important Points:

- The epsilon-delta definition is the rigorous foundation of calculus
- You don't need to find the "best" δ, just one that works
- δ depends on both ε and a
- Limits are unique (if they exist)
- This definition extends to limits at infinity and infinite limits

Practice Exercises

Problem 1: Domain of Functions

Determine the domain of the following functions:

(a) $f(x) = \dfrac{1}{x^2 - x}$

(b) $g(x) = \sqrt{x+2}$

(c) $h(x) = \dfrac{x}{x^2 - 3x}$

(d) $f(x) = \ln(x-1)$

Problem 2: Even and Odd Functions

Determine whether the following functions are even, odd, or neither:

(a) $f(x) = x^5 + x$

(b) $g(x) = x^4 - 2x^2 + 1$

(c) $h(x) = x^3 + x^2$

(d) $k(x) = \dfrac{x}{x^2 + 1}$

Problem 3: Function Composition

Let $f(x) = x^2 + 1$ and $g(x) = 2x - 3$.

(a) Find $(f \circ g)(x)$

(b) Find $(g \circ f)(x)$

(c) Evaluate $(f \circ g)(2)$

(d) Find the domain of $f \circ g$

Problem 4: Inverse Functions

Find the inverse function $f^{-1}(x)$ for each of the following:

(a) $f(x) = 3x - 7$

(b) $f(x) = x^3 + 1$

(c) $f(x) = \dfrac{2x+1}{x-3}$

Verify your answers by showing $f(f^{-1}(x)) = x$.

Problem 5: Circle and Line Equations

(a) Find an equation of a circle with center $(3, \frac{1}{2})$ and radius 5.

(b) Find the equation of the line that has slope $-\frac{5}{4}$ and y-intercept 6.

(c) Find the slope and y-intercept of the line $3x + 4y = 12$.

Problem 6: Basic Limit Evaluation

Evaluate the following limits:

(a) $\lim\limits_{x \to 2} \dfrac{x^3 + 2x - 12}{5 - x}$

(b) $\lim_{x \to 1} \frac{x^2 + x - 2}{x^2 - x}$

(c) $\lim_{x \to -3} \sqrt{\frac{x+2}{x+1}}$

(d) $\lim_{x \to 5} (2x^2 - 3x + 1)$

Problem 7: Limits with Rationalization

Evaluate the following limits:

(a) $\lim_{x \to 9} \frac{\sqrt{x} - 3}{x - 9}$

(b) $\lim_{x \to 2} \frac{4x - x^2}{2 - \sqrt{x}}$

(c) $\lim_{x \to 5} \frac{4 - x}{\sqrt{x+4} - \sqrt{x-4}}$

(d) $\lim_{x \to 0} \frac{\sqrt{1+x} - 1}{x}$

Problem 8: Trigonometric Limits

Evaluate the following limits:

(a) $\lim_{h \to 0} \frac{h}{\sin 3h}$

(b) $\lim_{x \to 0} \frac{\sin 2x}{5x}$

(c) $\lim_{\theta \to 0} \frac{1 - \cos\theta}{\theta^2}$

(d) $\lim_{x \to 0} \frac{\tan 4x}{x}$

(e) $\lim_{x \to 0} \frac{\sin 5x}{\sin 2x}$

Problem 9: Limits at Infinity

Evaluate the following limits:

(a) $\lim_{x \to \infty} \frac{3x^2 - 5x + 2}{2x^2 + x - 1}$

(b) $\lim_{x \to \infty} \frac{x^3 + 2x}{2x^2 - 5}$

(c) $\lim_{x \to \infty} \left(\sqrt{x^2 + 4x} - x\right)$

(d) $\lim_{t \to \infty} t^2 e^{-t}$

Problem 10: One-Sided Limits

For the function $f(x) = \begin{cases} x^2, & x < 2 \\ 3x - 2, & x \geq 2 \end{cases}$

(a) Find $\lim_{x \to 2^-} f(x)$

(b) Find $\lim_{x \to 2^+} f(x)$

(c) Does $\lim_{x \to 2} f(x)$ exist? Explain.

(d) Find $f(2)$.

Problem 11: Exponential and Logarithmic Limits

Evaluate the following limits:

(a) $\lim_{x \to 0} \frac{e^{3x} - 1}{x}$

(b) $\lim_{x \to 0} \frac{\ln(1 + 2x)}{x}$

(c) $\lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n$

(d) $\lim_{x \to 0} \frac{a^x - 1}{x}$ where $a > 0$

Problem 12: Complex Fraction Limits

Evaluate the following limits:

(a) $\lim_{h \to 0} \frac{\frac{1}{x+h} - \frac{1}{x}}{h}$

(b) $\lim_{x \to 1} \frac{\frac{1}{x} - 1}{x - 1}$

(c) $\lim_{h \to 0} \frac{(x + h)^2 - x^2}{h}$

Problem 13: Testing Continuity

Determine whether the following functions are continuous at the given point:

(a) $f(x) = \frac{x^2 - 4}{x - 2}$ at $x = 2$

(b) $g(x) = \begin{cases} x + 1, & x \leq 1 \\ x^2, & x > 1 \end{cases}$ at $x = 1$

(c) $h(x) = \frac{1}{x - 3}$ at $x = 3$

Problem 14: Piecewise Continuity (Type 1)

For what value of c is the function

$$f(x) = \begin{cases} x + 2, & x \leq -1 \\ c + 2, & x > -1 \end{cases}$$

continuous at every x? Verify that $\lim_{x \to -1} f(x)$ exists with your value of c.

Problem 15: Piecewise Continuity (Type 2)

For what value of a is

$$f(x) = \begin{cases} x^2 - 1, & x < 3 \\ 2ax, & x \geq 3 \end{cases}$$

continuous at every x?

Problem 16: Piecewise Continuity (Type 3)

For what value of b is

$$f(x) = \begin{cases} x, & x < -2 \\ bx^2, & x \geq -2 \end{cases}$$

continuous at every x?

Problem 17: Continuous Extension

Define $g(x)$ in a way that extends $g(x) = \dfrac{x^2 - 9}{x - 3}$ to be continuous at $x = 3$.

What should be the value of $g(3)$?

Problem 18: Domain of Continuity

At what points are the following functions continuous?

(a) $f(x) = \dfrac{1}{x^2 - 3x}$

(b) $g(x) = \dfrac{x + 1}{x^2 - 4}$

(c) $h(x) = \sqrt{x - 5}$

(d) $k(x) = \ln(x^2 - 1)$

Problem 19: Proving Limits (Epsilon-Delta)

Use the epsilon-delta definition to prove:

(a) $\lim_{x \to 2}(3x + 1) = 7$

(b) $\lim_{x \to 1} x^2 = 1$

(c) $\lim_{x \to 3}(2x - 5) = 1$

Problem 20: Limit Proof (Infinite)

Use the precise definition to prove that:

$$\lim_{x \to \infty} (x + 3)^3 = \infty$$

Find an appropriate N for any given $M > 0$.

Problem 21: Special Limit Proof

Show that:

$$\lim_{x \to 0} \frac{a^x - 1}{x} = \ln a$$

where $a > 0$ and $a \neq 1$. Use the fact that $a^x = e^{x \ln a}$.

Problem 22: Squeeze Theorem Application

Use the Squeeze Theorem to evaluate:

(a) $\lim_{x \to 0} x^2 \sin\left(\frac{1}{x}\right)$

(b) $\lim_{x \to 0} x^2 \cos\left(\frac{1}{x^2}\right)$

Hint: Use the fact that $-1 \leq \sin(u) \leq 1$ and $-1 \leq \cos(u) \leq 1$ for all u.

Problem 23: Limit Involving Absolute Value

Solve the inequality:

$$|x - 3| + |x + 2| < 11$$

Hint: Consider cases based on the sign of $(x - 3)$ and $(x + 2)$.

Problem 24: Hyperbolic Function

Solve the equation:

$$\frac{5}{4}e^x = \frac{1}{4}\sinh x$$

where $\sinh x = \dfrac{e^x - e^{-x}}{2}$.

Does this equation have a real solution? Justify your answer.

Problem 25: Continuity Proof

If f and g are both continuous at a, prove that:

(a) $f + g$ is continuous at a

(b) cf is continuous at a for any constant c

(c) $\dfrac{f}{g}$ is continuous at a, provided $g(a) \neq 0$

Use the definition of continuity and limit laws.

Chapter 2

Differential Calculus

Concept of Differentiation

Differentiation is one of the two fundamental operations in calculus (the other being integration). It measures how a function changes as its input changes. This concept has profound applications in physics, engineering, economics, and all quantitative sciences.

Definition 2.1: The Derivative (First Principles)

The **derivative** of a function f at a point $x = a$ is defined as:

$$\boxed{f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}}$$

provided this limit exists.

Alternative form:

$$\boxed{f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}}$$

Terminology:

- If the limit exists, we say f is **differentiable** at $x = a$
- $f'(a)$ is read as "f prime of a"
- The process of finding $f'(a)$ is called **differentiation**
- Other notations: $\left.\frac{df}{dx}\right|_{x=a}$, $\left.\frac{dy}{dx}\right|_{x=a}$, $Df(a)$

The Difference Quotient:

The expression $\frac{f(a+h) - f(a)}{h}$ is called the **difference quotient**. It represents:

- Numerator: Change in function value (rise)
- Denominator: Change in input value (run)
- Quotient: Average rate of change

Taking the limit as $h \to 0$ gives the **instantaneous rate of change**.

Definition 2.2: The Derivative as a Function

If f is differentiable at every point in its domain, we can define the **derivative function** $f'(x)$:

$$\boxed{f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}}$$

Common Notations:

$$f'(x) = \frac{df}{dx} = \frac{dy}{dx} = \frac{d}{dx}[f(x)] = Df(x) = D_x f(x)$$

Understanding the Notation:

- $\dfrac{dy}{dx}$: "dee y dee x" - Leibniz notation (most common)
- $f'(x)$: "f prime of x" - Lagrange notation
- Df: "D of f" - Operator notation
- The "d" represents an infinitesimally small change

Example 1: Finding Derivative from First Principles (Linear Function)

Problem: Use the definition of the derivative to find $f'(x)$ if $f(x) = 3x + 2$.

Solution:
Step 1: Write the definition

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Step 2: Find $f(x+h)$

$$\begin{aligned} f(x+h) &= 3(x+h) + 2 \\ &= 3x + 3h + 2 \end{aligned}$$

Step 3: Compute the difference quotient

$$\begin{aligned} \frac{f(x+h) - f(x)}{h} &= \frac{(3x + 3h + 2) - (3x + 2)}{h} \\ &= \frac{3x + 3h + 2 - 3x - 2}{h} \\ &= \frac{3h}{h} \\ &= 3 \quad (\text{for } h \neq 0) \end{aligned}$$

Step 4: Take the limit

$$\begin{aligned} f'(x) &= \lim_{h \to 0} 3 \\ &= 3 \end{aligned}$$

Answer: $\boxed{f'(x) = 3}$

Interpretation: The derivative is constant and equals 3, which is the slope of the line $y = 3x + 2$. This makes sense: linear functions have constant rate of change!

Example 2: Derivative from First Principles (Quadratic Function)

Problem: Use the definition to find $f'(x)$ if $f(x) = x^2$.

Solution:
Step 1: Write the definition

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Step 2: Find $f(x+h)$

$$\begin{aligned} f(x+h) &= (x+h)^2 \\ &= x^2 + 2xh + h^2 \end{aligned}$$

Step 3: Compute the difference quotient

$$\begin{aligned} \frac{f(x+h) - f(x)}{h} &= \frac{(x^2 + 2xh + h^2) - x^2}{h} \\ &= \frac{x^2 + 2xh + h^2 - x^2}{h} \\ &= \frac{2xh + h^2}{h} \\ &= \frac{h(2x+h)}{h} \\ &= 2x + h \quad (\text{for } h \neq 0) \end{aligned}$$

Step 4: Take the limit

$$\begin{aligned} f'(x) &= \lim_{h \to 0} (2x + h) \\ &= 2x + 0 \\ &= 2x \end{aligned}$$

Answer: $\boxed{f'(x) = 2x}$

Verification: At $x = 3$: $f'(3) = 2(3) = 6$. This means at the point $(3, 9)$ on $y = x^2$, the instantaneous rate of change is 6.

Pattern: Notice that the derivative of x^2 is $2x$. This is a special case of the power rule!

Example 3: Derivative from First Principles (Reciprocal Function)

Problem: Find $f'(x)$ if $f(x) = \dfrac{1}{x}$ using first principles.

Solution:
Step 1: Write the definition

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Step 2: Find $f(x+h)$

$$f(x+h) = \frac{1}{x+h}$$

Step 3: Compute the difference quotient

$$\frac{f(x+h)-f(x)}{h}=\frac{\frac{1}{x+h}-\frac{1}{x}}{h}$$

Simplify the complex fraction (find common denominator in numerator):

$$\begin{aligned}
&= \frac{\frac{x-(x+h)}{x(x+h)}}{h} \\
&= \frac{\frac{x-x-h}{x(x+h)}}{h} \\
&= \frac{\frac{-h}{x(x+h)}}{h} \\
&= \frac{-h}{x(x+h)}\cdot\frac{1}{h} \\
&= \frac{-h}{hx(x+h)} \\
&= \frac{-1}{x(x+h)} \quad (\text{for } h \neq 0)
\end{aligned}$$

Step 4: Take the limit

$$\begin{aligned}
f'(x) &= \lim_{h\to 0}\frac{-1}{x(x+h)} \\
&= \frac{-1}{x(x+0)} \\
&= \frac{-1}{x^2} \\
&= -\frac{1}{x^2}
\end{aligned}$$

Answer: $\boxed{f'(x)=-\dfrac{1}{x^2}}$

⚠ **Note:** The derivative is always negative, which makes sense because $f(x)=\frac{1}{x}$ is decreasing on $(0,\infty)$ and on $(-\infty,0)$.

Example 4: Derivative from First Principles (Square Root)

Problem: Find $f'(x)$ if $f(x)=\sqrt{x}$ using the definition.

Solution:

Step 1: Write the definition

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

Step 2: Set up the difference quotient

$$f'(x)=\lim_{h\to 0}\frac{\sqrt{x+h}-\sqrt{x}}{h}$$

This gives $\frac{0}{0}$ form. Use rationalization!

Step 3: Rationalize the numerator
Multiply by conjugate:

$$\begin{aligned}
&= \lim_{h\to 0} \frac{\sqrt{x+h}-\sqrt{x}}{h} \cdot \frac{\sqrt{x+h}+\sqrt{x}}{\sqrt{x+h}+\sqrt{x}} \\
&= \lim_{h\to 0} \frac{(\sqrt{x+h})^2-(\sqrt{x})^2}{h(\sqrt{x+h}+\sqrt{x})} \\
&= \lim_{h\to 0} \frac{(x+h)-x}{h(\sqrt{x+h}+\sqrt{x})} \\
&= \lim_{h\to 0} \frac{h}{h(\sqrt{x+h}+\sqrt{x})} \\
&= \lim_{h\to 0} \frac{1}{\sqrt{x+h}+\sqrt{x}}
\end{aligned}$$

Step 4: Evaluate the limit

$$\begin{aligned}
f'(x) &= \frac{1}{\sqrt{x+0}+\sqrt{x}} \\
&= \frac{1}{\sqrt{x}+\sqrt{x}} \\
&= \frac{1}{2\sqrt{x}}
\end{aligned}$$

Answer: $\boxed{f'(x) = \frac{1}{2\sqrt{x}}}$

Verification: This can also be written as $f'(x) = \frac{1}{2}x^{-1/2}$, which matches the power rule: $\frac{d}{dx}(x^{1/2}) = \frac{1}{2}x^{-1/2}$ ✓

⚠ **Domain Note:** $f'(x)$ is defined only for $x > 0$ (same as domain of $\sqrt{x}$).

Definition 2.3: Differentiability

A function f is **differentiable at** $x = a$ if $f'(a)$ exists (i.e., the limit defining the derivative exists and is finite).

A function is **differentiable on an interval** if it is differentiable at every point in that interval.

Important Relationship:

Theorem: If f is differentiable at $x = a$, then f is continuous at $x = a$.

Warning: The converse is NOT true! A function can be continuous but not differentiable.

Example: $f(x) = |x|$ is continuous at $x = 0$ but not differentiable there (sharp corner).

When is a function NOT differentiable?

- At a corner or cusp (like $|x|$ at $x = 0$)
- At a discontinuity
- At a vertical tangent line
- Where the function is not defined

Example 5: Testing Differentiability

Problem: Determine if $f(x) = |x|$ is differentiable at $x = 0$.

Solution:

We need to check if $\lim_{h\to 0} \frac{f(0+h) - f(0)}{h}$ exists.

Step 1: Find $f(0)$

$$f(0) = |0| = 0$$

Step 2: Check the right-hand derivative

For $h > 0$:

$$\begin{aligned}
\lim_{h\to 0^+} \frac{f(0+h) - f(0)}{h} &= \lim_{h\to 0^+} \frac{|h| - 0}{h} \\
&= \lim_{h\to 0^+} \frac{h}{h} \quad (\text{since } h > 0, |h| = h) \\
&= \lim_{h\to 0^+} 1 \\
&= 1
\end{aligned}$$

Step 3: Check the left-hand derivative

For $h < 0$:

$$\begin{aligned}
\lim_{h\to 0^-} \frac{f(0+h) - f(0)}{h} &= \lim_{h\to 0^-} \frac{|h| - 0}{h} \\
&= \lim_{h\to 0^-} \frac{-h}{h} \quad (\text{since } h < 0, |h| = -h) \\
&= \lim_{h\to 0^-} (-1) \\
&= -1
\end{aligned}$$

Step 4: Compare one-sided limits

Right-hand derivative: 1

Left-hand derivative: -1

Since $1 \neq -1$, the two-sided limit does not exist.

Conclusion: $f(x) = |x|$ is **NOT differentiable** at $x = 0$.

Answer: $\boxed{\text{Not differentiable at } x = 0}$

Geometric Interpretation: The graph of $|x|$ has a sharp corner at $x = 0$, so there's no unique tangent line there. The "slope from the right" is 1, and the "slope from the left" is -1.

Section 2.1 Summary

The Derivative Definition:

$$\boxed{f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}}$$

Steps for Computing Derivatives from First Principles:

1. Find $f(x+h)$ by substituting $x+h$ into the function
2. Form the difference quotient: $\frac{f(x+h) - f(x)}{h}$

3. Simplify algebraically (factor, rationalize, combine fractions)

4. Take the limit as $h \to 0$

Key Derivatives from First Principles:

Function $f(x)$	**Derivative** $f'(x)$
c (constant)	0
x	1
x^2	$2x$
x^n	nx^{n-1}
$\frac{1}{x}$	$-\frac{1}{x^2}$
$\sqrt{x}$	$\frac{1}{2\sqrt{x}}$

Important Facts:

- Differentiable $\Longrightarrow$ Continuous
- Continuous $\not\Longrightarrow$ Differentiable
- Not differentiable at: corners, cusps, discontinuities, vertical tangents

Geometrical and Physical Meaning of Derivatives

The derivative has profound interpretations in both geometry and physics. Understanding these interpretations helps us apply calculus to real-world problems and develop intuition for what derivatives tell us about functions.

Definition 2.4: Geometric Interpretation - Slope of Tangent Line

The derivative $f'(a)$ represents the **slope of the tangent line** to the graph of $y = f(x)$ at the point $(a, f(a))$.

Tangent Line Equation:
The equation of the tangent line to $y = f(x)$ at $x = a$ is:

$$\boxed{y - f(a) = f'(a)(x - a)}$$

or in slope-intercept form:

$$\boxed{y = f'(a)(x - a) + f(a)}$$

Understanding the Tangent Line:

- The tangent line is the best linear approximation to the curve at that point
- It "just touches" the curve at $(a, f(a))$
- The slope $m = f'(a)$ tells us the direction and steepness
- If $f'(a) > 0$: function is increasing at $x = a$ (upward slope)
- If $f'(a) < 0$: function is decreasing at $x = a$ (downward slope)
- If $f'(a) = 0$: horizontal tangent (possible max/min/inflection)

Example 6: Finding Tangent Line Equation

Problem: Find the equation of the tangent line to $y = x^2$ at the point $(2, 4)$.

Solution:
Step 1: Find the derivative
From Example 2, we know $f'(x) = 2x$.
Step 2: Find the slope at $x = 2$

$$f'(2) = 2(2) = 4$$

So the slope of the tangent line is $m = 4$.
Step 3: Use point-slope form
The tangent line passes through $(2, 4)$ with slope 4:

$$\begin{aligned} y - 4 &= 4(x - 2) \\ y - 4 &= 4x - 8 \\ y &= 4x - 4 \end{aligned}$$

Answer: $\boxed{y = 4x - 4}$
Verification:

- Does it pass through $(2, 4)$? $y = 4(2) - 4 = 4$ ✓
- Does it have slope 4? Yes ✓

Visualization: At $x = 2$, the parabola $y = x^2$ is rising with slope 4. The tangent line $y = 4x - 4$ has the same slope and touches the parabola at exactly $(2, 4)$.

Definition 2.5: Normal Line

The **normal line** to a curve at a point is the line perpendicular to the tangent line at that point.

Normal Line Equation:

If the tangent line has slope $m = f'(a)$, the normal line has slope $m_{\text{normal}} = -\dfrac{1}{f'(a)}$ (negative reciprocal).

The equation of the normal line at $(a, f(a))$ is:

$$\boxed{y - f(a) = -\frac{1}{f'(a)}(x - a)}$$

provided $f'(a) \neq 0$.

Perpendicular Slopes:

Two lines are perpendicular if and only if the product of their slopes is -1:

$$m_1 \cdot m_2 = -1 \iff m_2 = -\frac{1}{m_1}$$

Special Cases:

- If $f'(a) = 0$ (horizontal tangent), the normal line is vertical: $x = a$
- If tangent line is vertical, the normal line is horizontal

Example 7: Finding Normal Line Equation

Problem: Find the equation of the normal line to the curve $y = x^2 - 3x + 2$ at $x = 1$. (Related to 2025 past paper, Q1 part viii)

Solution:

Step 1: Find the point on the curve

When $x = 1$:

$$y = 1^2 - 3(1) + 2 = 1 - 3 + 2 = 0$$

So the point is $(1, 0)$.

Step 2: Find the derivative

$$f(x) = x^2 - 3x + 2$$
$$f'(x) = 2x - 3$$

Step 3: Find the slope of the tangent line at $x = 1$

$$f'(1) = 2(1) - 3 = 2 - 3 = -1$$

Step 4: Find the slope of the normal line

The normal line is perpendicular to the tangent, so:

$$m_{\text{normal}} = -\frac{1}{f'(1)} = -\frac{1}{-1} = 1$$

Step 5: Write the equation of the normal line
Using point-slope form with point $(1, 0)$ and slope 1:

$$\begin{aligned} y - 0 &= 1(x - 1) \\ y &= x - 1 \end{aligned}$$

Answer: $\boxed{y = x - 1}$
Check: The tangent line has slope -1 and the normal line has slope 1. Product: $(-1)(1) = -1$ ✓ (perpendicular!)

Example 8: Horizontal Tangent Lines

Problem: Find the point(s) on the curve $y = x^4 - 6x^2 + 4$ where the tangent line is horizontal.
(From 2025 past paper, Q1 part iii)

Solution:
The tangent line is horizontal when $f'(x) = 0$.
Step 1: Find the derivative

$$\begin{aligned} f(x) &= x^4 - 6x^2 + 4 \\ f'(x) &= 4x^3 - 12x \end{aligned}$$

Step 2: Set the derivative equal to zero

$$\begin{aligned} 4x^3 - 12x &= 0 \\ 4x(x^2 - 3) &= 0 \\ 4x(x - \sqrt{3})(x + \sqrt{3}) &= 0 \end{aligned}$$

Solutions: $x = 0$, $x = \sqrt{3}$, $x = -\sqrt{3}$
Step 3: Find the corresponding y-coordinates
For $x = 0$:

$$y = 0^4 - 6(0)^2 + 4 = 4$$

For $x = \sqrt{3}$:

$$y = (\sqrt{3})^4 - 6(\sqrt{3})^2 + 4 = 9 - 18 + 4 = -5$$

For $x = -\sqrt{3}$:

$$y = (-\sqrt{3})^4 - 6(-\sqrt{3})^2 + 4 = 9 - 18 + 4 = -5$$

Answer: The tangent line is horizontal at three points:

$$\boxed{(0, 4), \quad (\sqrt{3}, -5), \quad (-\sqrt{3}, -5)}$$

Significance: These are critical points where the function may have local maxima or minima.

Definition 2.6: Physical Interpretation - Rate of Change

In physics and applied sciences, the derivative represents the **instantaneous rate of change**.

Common Physical Interpretations:

1. Velocity (Physics):

If $s(t)$ is the position of an object at time t, then:

$$\boxed{v(t) = s'(t) = \frac{ds}{dt}}$$

is the **velocity** (instantaneous rate of change of position).

2. Acceleration (Physics):

If $v(t)$ is the velocity at time t, then:

$$\boxed{a(t) = v'(t) = \frac{dv}{dt} = s''(t) = \frac{d^2s}{dt^2}}$$

is the **acceleration** (instantaneous rate of change of velocity).

3. General Rate of Change:

For any quantity $Q(t)$ depending on time:

$$\boxed{\frac{dQ}{dt} = \text{instantaneous rate of change of } Q}$$

Sign Interpretation:

- $v > 0$: moving forward (in positive direction)
- $v < 0$: moving backward (in negative direction)
- $v = 0$: instantaneously at rest
- $a > 0$: speeding up (if $v > 0$) or slowing down (if $v < 0$)
- $a < 0$: slowing down (if $v > 0$) or speeding up (if $v < 0$)

Example 9: Velocity and Position

Problem: The position of a particle moving along a line is given by $s(t) = t^3 - 6t^2 + 9t$ meters, where t is in seconds.

(a) Find the velocity at time t.

(b) When is the particle at rest?

(c) Find the velocity at $t = 2$ seconds.

Solution:

(a) Find velocity function

Velocity is the derivative of position:

$$\begin{aligned} v(t) &= s'(t) \\ &= \frac{d}{dt}(t^3 - 6t^2 + 9t) \\ &= 3t^2 - 12t + 9 \end{aligned}$$

Answer: $\boxed{v(t) = 3t^2 - 12t + 9 \text{ m/s}}$

(b) When is the particle at rest?
The particle is at rest when $v(t) = 0$:

$$\begin{aligned} 3t^2 - 12t + 9 &= 0 \\ 3(t^2 - 4t + 3) &= 0 \\ 3(t-1)(t-3) &= 0 \end{aligned}$$

Solutions: $t = 1$ or $t = 3$
Answer: The particle is at rest at $\boxed{t = 1 \text{ s and } t = 3 \text{ s}}$

(c) Velocity at $t = 2$ seconds

$$\begin{aligned} v(2) &= 3(2)^2 - 12(2) + 9 \\ &= 12 - 24 + 9 \\ &= -3 \end{aligned}$$

Answer: $\boxed{v(2) = -3 \text{ m/s}}$

Interpretation: At $t = 2$ seconds, the particle is moving backward (negative velocity) at 3 m/s.

Example 10: Velocity from Given Equation

Problem: The velocity of a moving object is given by $v(t) = 4t + 2$ (in meters per second). Find the total distance travelled from $t = 0$ to $t = 5$ seconds.

(From 2025 past paper, Q1 part ix - concept question)

Solution:

Understanding: Distance travelled is related to the integral of velocity. However, since we haven't covered integration yet, we interpret this from the derivative perspective.

Step 1: Check if velocity changes sign
$v(t) = 4t + 2$
For $t \geq 0$: $v(t) = 4t + 2 \geq 2 > 0$
The velocity is always positive, so the object always moves forward.

Step 2: Distance calculation concept
When velocity is always positive (or always negative), the distance equals the absolute value of displacement.

Distance $= \int_0^5 v(t)\,dt = \int_0^5 (4t+2)\,dt$
Using the antiderivative (which we'll learn in integration):

$$\begin{aligned} \text{Distance} &= \left[2t^2 + 2t\right]_0^5 \\ &= [2(5)^2 + 2(5)] - [0] \\ &= 50 + 10 \\ &= 60 \end{aligned}$$

Answer: $\boxed{60 \text{ meters}}$

Note: This problem connects derivatives (velocity) to integrals (distance), which we'll formalize in Chapter 5.

Definition 2.7: Average Rate of Change vs. Instantaneous Rate of Change

Average Rate of Change:

The average rate of change of $f(x)$ over the interval $[a, b]$ is:

$$\boxed{\text{Average Rate} = \frac{f(b) - f(a)}{b - a}}$$

This is the slope of the **secant line** connecting $(a, f(a))$ and $(b, f(b))$.

Instantaneous Rate of Change:

The instantaneous rate of change at $x = a$ is:

$$\boxed{\text{Instantaneous Rate} = f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}}$$

This is the slope of the **tangent line** at $x = a$.

Relationship:

The instantaneous rate of change is the limit of average rates of change as the interval shrinks to a point:

$$f'(a) = \lim_{b \to a} \frac{f(b) - f(a)}{b - a}$$

Example 11: Average vs. Instantaneous Rate of Change

Problem: For $f(x) = x^2$:

(a) Find the average rate of change from $x = 1$ to $x = 3$.

(b) Find the instantaneous rate of change at $x = 1$.

(c) Find the instantaneous rate of change at $x = 3$.

Solution:

(a) Average rate of change from $x = 1$ to $x = 3$

$$\begin{aligned}\text{Average Rate} &= \frac{f(3) - f(1)}{3 - 1} \\ &= \frac{9 - 1}{2} \\ &= \frac{8}{2} \\ &= 4\end{aligned}$$

Answer: $\boxed{4}$

(b) Instantaneous rate at $x = 1$

We know $f'(x) = 2x$, so:

$$f'(1) = 2(1) = 2$$

Answer: $\boxed{2}$

(c) Instantaneous rate at $x = 3$

$$f'(3) = 2(3) = 6$$

Answer: $\boxed{6}$

Observation: The average rate of change (4) lies between the instantaneous rates at the endpoints (2 and 6). This is related to the Mean Value Theorem!

Section 2.2 Summary

Geometric Interpretations:

Concept	**Meaning**
$f'(a)$	Slope of tangent line at $(a, f(a))$
Tangent Line	$y - f(a) = f'(a)(x - a)$
Normal Line	$y - f(a) = -\frac{1}{f'(a)}(x - a)$ (perpendicular to tangent)
$f'(a) = 0$	Horizontal tangent (possible max/min)
$f'(a) > 0$	Function increasing at $x = a$
$f'(a) < 0$	Function decreasing at $x = a$

Physical Interpretations:

If $s(t)$ is...	**Then...**
Position	$v(t) = s'(t)$ is velocity
Velocity	$a(t) = v'(t) = s''(t)$ is acceleration
Any quantity $Q(t)$	$Q'(t)$ is rate of change of Q

Key Formulas to Remember:

- Tangent line: $y = f'(a)(x - a) + f(a)$
- Normal line slope: $m_n = -\dfrac{1}{f'(a)}$
- Velocity: $v = \dfrac{ds}{dt}$
- Acceleration: $a = \dfrac{dv}{dt} = \dfrac{d^2s}{dt^2}$
- Average rate: $\dfrac{f(b) - f(a)}{b - a}$
- Instantaneous rate: $f'(a)$

Common Mistakes:

- Confusing tangent line with normal line
- Forgetting negative reciprocal for perpendicular slopes
- Mixing up position, velocity, and acceleration
- Confusing average and instantaneous rates of change

Basic Rules of Differentiation

Computing derivatives from first principles every time is tedious. Fortunately, we can develop general rules that allow us to differentiate functions quickly and efficiently. These rules form the foundation of differential calculus.

Theorem 2.1: The Power Rule

If $f(x) = x^n$ where n is any real number, then:

$$\boxed{\frac{d}{dx}(x^n) = nx^{n-1}}$$

Special Cases:

- If $f(x) = c$ (constant), then $f'(x) = 0$
- If $f(x) = x$, then $f'(x) = 1$
- If $f(x) = x^2$, then $f'(x) = 2x$
- If $f(x) = x^3$, then $f'(x) = 3x^2$

Proof of Power Rule (for positive integer n)

Proof:

We prove this for n a positive integer using the binomial theorem.
Let $f(x) = x^n$. By definition:

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h\to 0} \frac{(x+h)^n - x^n}{h} \end{aligned}$$

Step 1: Expand $(x+h)^n$ using the binomial theorem:

$$(x+h)^n = x^n + nx^{n-1}h + \frac{n(n-1)}{2}x^{n-2}h^2 + \cdots + h^n$$

Step 2: Substitute into the limit:

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{x^n + nx^{n-1}h + \frac{n(n-1)}{2}x^{n-2}h^2 + \cdots + h^n - x^n}{h} \\ &= \lim_{h\to 0} \frac{nx^{n-1}h + \frac{n(n-1)}{2}x^{n-2}h^2 + \cdots + h^n}{h} \end{aligned}$$

Step 3: Factor out h from numerator:

$$\begin{aligned} &= \lim_{h\to 0} \frac{h\left(nx^{n-1} + \frac{n(n-1)}{2}x^{n-2}h + \cdots + h^{n-1}\right)}{h} \\ &= \lim_{h\to 0} \left(nx^{n-1} + \frac{n(n-1)}{2}x^{n-2}h + \cdots + h^{n-1}\right) \end{aligned}$$

Step 4: Evaluate the limit as $h \to 0$:

All terms with h vanish:

$$f'(x) = nx^{n-1}$$

Conclusion: For positive integer n, $\frac{d}{dx}(x^n) = nx^{n-1}$. □

⚠ **Note:** The proof can be extended to negative integers, rational numbers, and all real numbers using more advanced techniques.

Theorem 2.2: Constant Multiple Rule

If c is a constant and f is differentiable, then:

$$\boxed{\frac{d}{dx}[cf(x)] = c \cdot f'(x)}$$

In words: The derivative of a constant times a function is the constant times the derivative of the function.

Theorem 2.3: Sum and Difference Rules

If f and g are differentiable functions, then:

Sum Rule:

$$\boxed{\frac{d}{dx}[f(x) + g(x)] = f'(x) + g'(x)}$$

Difference Rule:

$$\boxed{\frac{d}{dx}[f(x) - g(x)] = f'(x) - g'(x)}$$

In words: The derivative of a sum (or difference) is the sum (or difference) of the derivatives.

Example 12: Using the Power Rule

Problem: Find the derivative of each function:

(a) $f(x) = x^5$
(b) $g(x) = x^{10}$
(c) $h(x) = x^{-3}$
(d) $k(x) = \sqrt[3]{x}$

Solution:

(a) $f(x) = x^5$
Using the power rule with $n = 5$:

$$f'(x) = 5x^{5-1} = 5x^4$$

Answer: $\boxed{f'(x) = 5x^4}$

(b) $g(x) = x^{10}$
Using the power rule with $n = 10$:

$$g'(x) = 10x^{10-1} = 10x^9$$

Answer: $\boxed{g'(x) = 10x^9}$

(c) $h(x) = x^{-3}$
Using the power rule with $n = -3$:

$$h'(x) = -3x^{-3-1} = -3x^{-4} = -\frac{3}{x^4}$$

Answer: $\boxed{h'(x) = -\frac{3}{x^4}}$

(d) $k(x) = \sqrt[3]{x} = x^{1/3}$
Using the power rule with $n = \frac{1}{3}$:

$$k'(x) = \frac{1}{3}x^{1/3-1} = \frac{1}{3}x^{-2/3} = \frac{1}{3x^{2/3}} = \frac{1}{3\sqrt[3]{x^2}}$$

Answer: $\boxed{k'(x) = \frac{1}{3\sqrt[3]{x^2}}}$

Pattern: Always reduce the exponent by 1 and multiply by the original exponent!

Example 13: Combining Basic Rules

Problem: Find the derivative of:
(a) $f(x) = 3x^4 + 2x^2 - 5x + 7$
(b) $g(x) = 4x^3 - 6x^2 + 2$
(c) $h(x) = \frac{2}{x^3} + 5\sqrt{x}$

Solution:
(a) $f(x) = 3x^4 + 2x^2 - 5x + 7$
Apply sum/difference and constant multiple rules:

$$\begin{aligned} f'(x) &= \frac{d}{dx}(3x^4) + \frac{d}{dx}(2x^2) - \frac{d}{dx}(5x) + \frac{d}{dx}(7) \\ &= 3 \cdot 4x^3 + 2 \cdot 2x - 5 \cdot 1 + 0 \\ &= 12x^3 + 4x - 5 \end{aligned}$$

Answer: $\boxed{f'(x) = 12x^3 + 4x - 5}$

(b) $g(x) = 4x^3 - 6x^2 + 2$

$$\begin{aligned} g'(x) &= \frac{d}{dx}(4x^3) - \frac{d}{dx}(6x^2) + \frac{d}{dx}(2) \\ &= 4 \cdot 3x^2 - 6 \cdot 2x + 0 \\ &= 12x^2 - 12x \end{aligned}$$

Answer: $\boxed{g'(x) = 12x^2 - 12x}$

(c) $h(x) = \frac{2}{x^3} + 5\sqrt{x}$

First rewrite using exponents:

$$h(x) = 2x^{-3} + 5x^{1/2}$$

Now differentiate:

$$\begin{aligned} h'(x) &= 2 \cdot (-3)x^{-4} + 5 \cdot \frac{1}{2}x^{-1/2} \\ &= -6x^{-4} + \frac{5}{2}x^{-1/2} \\ &= -\frac{6}{x^4} + \frac{5}{2\sqrt{x}} \end{aligned}$$

Answer: $\boxed{h'(x) = -\frac{6}{x^4} + \frac{5}{2\sqrt{x}}}$

⚠ **Key Technique:** Always rewrite roots and fractions as powers before differentiating!

Example 14: Derivative Practice

Problem: Find $\frac{dy}{dx}$ for:

(a) $y = x^5 + x$

(b) $y = \frac{x+1}{x-2}$ (we'll need quotient rule later, but try expanding first if possible)

(c) $y = 2x^3 - 3x^2 - 12x + 1$

Solution:

(a) $y = x^5 + x$

$$\frac{dy}{dx} = 5x^4 + 1$$

Answer: $\boxed{\frac{dy}{dx} = 5x^4 + 1}$

(b) $y = \frac{x+1}{x-2}$

For this, we'll need the quotient rule (covered in Section 2.4). Let's note it for now.

Answer: (Requires quotient rule - see next section)

(c) $y = 2x^3 - 3x^2 - 12x + 1$

$$\begin{aligned} \frac{dy}{dx} &= 2(3x^2) - 3(2x) - 12(1) + 0 \\ &= 6x^2 - 6x - 12 \end{aligned}$$

Answer: $\boxed{\frac{dy}{dx} = 6x^2 - 6x - 12}$

Note: This is related to past paper 2024 Q4(a) where we need to find inflection points!

Theorem 2.4: Derivatives of Trigonometric Functions

The derivatives of the basic trigonometric functions are:

$$\boxed{\frac{d}{dx}(\sin x) = \cos x}$$

$$\boxed{\frac{d}{dx}(\cos x) = -\sin x}$$

$$\boxed{\frac{d}{dx}(\tan x) = \sec^2 x}$$

$$\boxed{\frac{d}{dx}(\cot x) = -\csc^2 x}$$

$$\boxed{\frac{d}{dx}(\sec x) = \sec x \tan x}$$

$$\boxed{\frac{d}{dx}(\csc x) = -\csc x \cot x}$$

⚠ **Important:** x must be in radians for these formulas to be valid!

Proof: Derivative of $\sin x$

Proof:

We prove $\frac{d}{dx}(\sin x) = \cos x$ using first principles and trigonometric identities.

Step 1: Write the definition

$$\frac{d}{dx}(\sin x) = \lim_{h\to 0} \frac{\sin(x+h) - \sin x}{h}$$

Step 2: Use the sine addition formula

Recall: $\sin(x+h) = \sin x \cos h + \cos x \sin h$

$$\begin{aligned}
\frac{d}{dx}(\sin x) &= \lim_{h\to 0} \frac{\sin x \cos h + \cos x \sin h - \sin x}{h} \\
&= \lim_{h\to 0} \frac{\sin x(\cos h - 1) + \cos x \sin h}{h} \\
&= \lim_{h\to 0} \left[\sin x \cdot \frac{\cos h - 1}{h} + \cos x \cdot \frac{\sin h}{h}\right]
\end{aligned}$$

Step 3: Separate the limits

$$= \sin x \cdot \lim_{h\to 0} \frac{\cos h - 1}{h} + \cos x \cdot \lim_{h\to 0} \frac{\sin h}{h}$$

Step 4: Evaluate using known limits

From Chapter 1, we know:

- $\lim_{h\to 0} \frac{\sin h}{h} = 1$

- $\lim_{h\to 0} \frac{\cos h - 1}{h} = 0$ (equivalent to $\lim_{h\to 0} \frac{1-\cos h}{h} = 0$)

Therefore:

$$\frac{d}{dx}(\sin x) = \sin x \cdot 0 + \cos x \cdot 1 = \cos x$$

Conclusion: $\frac{d}{dx}(\sin x) = \cos x$ □

Example 15: Derivatives of Trigonometric Functions

Problem: Find the derivative:

(a) $f(x) = 3\sin x + 2\cos x$
(b) $g(x) = x^2 + \tan x$
(c) $h(x) = 5\sin x - 4\cos x + x^3$

Solution:

(a) $f(x) = 3\sin x + 2\cos x$

$$\begin{aligned} f'(x) &= 3 \cdot \frac{d}{dx}(\sin x) + 2 \cdot \frac{d}{dx}(\cos x) \\ &= 3\cos x + 2(-\sin x) \\ &= 3\cos x - 2\sin x \end{aligned}$$

Answer: $\boxed{f'(x) = 3\cos x - 2\sin x}$

(b) $g(x) = x^2 + \tan x$

$$\begin{aligned} g'(x) &= \frac{d}{dx}(x^2) + \frac{d}{dx}(\tan x) \\ &= 2x + \sec^2 x \end{aligned}$$

Answer: $\boxed{g'(x) = 2x + \sec^2 x}$

(c) $h(x) = 5\sin x - 4\cos x + x^3$

$$\begin{aligned} h'(x) &= 5\cos x - 4(-\sin x) + 3x^2 \\ &= 5\cos x + 4\sin x + 3x^2 \end{aligned}$$

Answer: $\boxed{h'(x) = 5\cos x + 4\sin x + 3x^2}$

Theorem 2.5: Derivatives of Exponential and Logarithmic Functions

Exponential Functions:

$$\boxed{\frac{d}{dx}(e^x) = e^x}$$

$$\boxed{\frac{d}{dx}(a^x) = a^x \ln a \quad (a > 0, a \neq 1)}$$

Logarithmic Functions:

$$\boxed{\frac{d}{dx}(\ln x) = \frac{1}{x} \quad (x > 0)}$$

$$\boxed{\frac{d}{dx}(\log_a x) = \frac{1}{x \ln a} \quad (x > 0, a > 0, a \neq 1)}$$

Special Property:

The function e^x is its own derivative! This makes e the natural base for exponential functions.

Example 16: Exponential and Logarithmic Derivatives

Problem: Find the derivative:

(a) $f(x) = 3e^x + x^2$
(b) $g(x) = \ln x + 2x$
(c) $h(x) = e^x + \ln x - 5x + 4$

Solution:

(a) $f(x) = 3e^x + x^2$

$$\begin{aligned} f'(x) &= 3 \cdot \frac{d}{dx}(e^x) + \frac{d}{dx}(x^2) \\ &= 3e^x + 2x \end{aligned}$$

Answer: $\boxed{f'(x) = 3e^x + 2x}$

(b) $g(x) = \ln x + 2x$

$$\begin{aligned} g'(x) &= \frac{d}{dx}(\ln x) + 2 \cdot \frac{d}{dx}(x) \\ &= \frac{1}{x} + 2 \end{aligned}$$

Answer: $\boxed{g'(x) = \frac{1}{x} + 2}$

(c) $h(x) = e^x + \ln x - 5x + 4$

$$h'(x) = e^x + \frac{1}{x} - 5$$

Answer: $\boxed{h'(x) = e^x + \frac{1}{x} - 5}$

Section 2.3 Summary: Basic Derivative Rules

Function	Derivative	Rule Name
c (constant)	0	Constant Rule
x^n	nx^{n-1}	Power Rule
$cf(x)$	$cf'(x)$	Constant Multiple
$f(x) \pm g(x)$	$f'(x) \pm g'(x)$	Sum/Difference
$\sin x$	$\cos x$	Trig
$\cos x$	$-\sin x$	Trig
$\tan x$	$\sec^2 x$	Trig
e^x	e^x	Exponential
$\ln x$	$\frac{1}{x}$	Logarithmic

Differentiation Strategy:

1. Rewrite roots and fractions as powers
2. Apply sum/difference rule to break up the function
3. Apply constant multiple rule
4. Apply power rule, trig rules, or exp/log rules to each term
5. Simplify your answer

Common Mistakes:

- Forgetting the negative sign in $\frac{d}{dx}(\cos x) = -\sin x$
- Writing $\frac{d}{dx}(x^n) = x^{n-1}$ (forgot to multiply by n!)
- Using degrees instead of radians for trig functions
- Forgetting domain restrictions (e.g., $\ln x$ only for $x > 0$)

Product and Quotient Rules

When functions are multiplied or divided, we cannot simply multiply or divide their derivatives. We need special rules for these operations.

Theorem 2.6: The Product Rule

If f and g are differentiable functions, then:

$$\boxed{\frac{d}{dx}[f(x) \cdot g(x)] = f(x) \cdot g'(x) + g(x) \cdot f'(x)}$$

Alternative notation:

$$(fg)' = fg' + gf'$$

or

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$$

In Words:

The derivative of a product is: (first function)(derivative of second) + (second function)(derivative of first).

Common Error:

$(fg)' \neq f'g'$! The derivative of a product is NOT the product of derivatives!

Proof of the Product Rule

Proof:

Let $h(x) = f(x) \cdot g(x)$. We use the definition of the derivative.

Step 1: Write the definition

$$h'(x) = \lim_{h \to 0} \frac{h(x+h) - h(x)}{h}$$

Step 2: Substitute $h(x) = f(x)g(x)$

$$h'(x) = \lim_{h \to 0} \frac{f(x+h)g(x+h) - f(x)g(x)}{h}$$

Step 3: Add and subtract $f(x+h)g(x)$ (clever trick!)

$$h'(x) = \lim_{h \to 0} \frac{f(x+h)g(x+h) - f(x+h)g(x) + f(x+h)g(x) - f(x)g(x)}{h}$$

Step 4: Factor and separate

$$\begin{aligned} &= \lim_{h \to 0} \frac{f(x+h)[g(x+h) - g(x)] + g(x)[f(x+h) - f(x)]}{h} \\ &= \lim_{h \to 0} \left[f(x+h) \cdot \frac{g(x+h) - g(x)}{h} + g(x) \cdot \frac{f(x+h) - f(x)}{h} \right] \end{aligned}$$

Step 5: Evaluate the limits

Since f is differentiable, it's continuous, so $\lim_{h \to 0} f(x+h) = f(x)$.

$$\begin{aligned} h'(x) &= \lim_{h \to 0} f(x+h) \cdot \lim_{h \to 0} \frac{g(x+h) - g(x)}{h} + g(x) \cdot \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \\ &= f(x) \cdot g'(x) + g(x) \cdot f'(x) \end{aligned}$$

Conclusion: $(fg)' = fg' + gf'$ □

Example 17: Using the Product Rule

Problem: Find the derivative of $f(x) = x^2 \sin x$.

Solution:
This is a product of two functions: $f(x) = x^2$ and $g(x) = \sin x$.
Step 1: Identify the parts

$$u = x^2 \implies u' = 2x$$
$$v = \sin x \implies v' = \cos x$$

Step 2: Apply the product rule
$(uv)' = uv' + vu'$

$$\begin{aligned}\frac{d}{dx}(x^2 \sin x) &= x^2 \cdot \frac{d}{dx}(\sin x) + \sin x \cdot \frac{d}{dx}(x^2) \\ &= x^2 \cdot \cos x + \sin x \cdot 2x \\ &= x^2 \cos x + 2x \sin x\end{aligned}$$

Answer: $\boxed{f'(x) = x^2 \cos x + 2x \sin x}$
Check: We can factor: $f'(x) = x(x \cos x + 2 \sin x)$ if needed.

Example 18: Product Rule with Exponential

Problem: Find $\frac{dy}{dx}$ if $y = e^x \sin x$.

Solution:
Step 1: Identify the factors

$$u = e^x \implies u' = e^x$$
$$v = \sin x \implies v' = \cos x$$

Step 2: Apply product rule

$$\begin{aligned}\frac{dy}{dx} &= e^x \cdot \cos x + \sin x \cdot e^x \\ &= e^x \cos x + e^x \sin x \\ &= e^x(\cos x + \sin x)\end{aligned}$$

Answer: $\boxed{\frac{dy}{dx} = e^x(\cos x + \sin x)}$
Note: We factored out e^x for a cleaner form.

Theorem 2.7: The Quotient Rule

If f and g are differentiable functions and $g(x) \neq 0$, then:

$$\boxed{\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x) \cdot f'(x) - f(x) \cdot g'(x)}{[g(x)]^2}}$$

Alternative notation:

$$\left(\frac{u}{v}\right)' = \frac{vu' - uv'}{v^2}$$

Memory Aid - "Lo-dee-hi minus hi-dee-lo over lo-lo":

- Lo (bottom) times dee-hi (derivative of top)
- Minus
- Hi (top) times dee-lo (derivative of bottom)
- All over
- Lo-lo (bottom squared)

Order Matters!
The order in the numerator is crucial: $vu' - uv'$, NOT $uv' - vu'$!

Example 19: Using the Quotient Rule

Problem: Find the derivative of $f(x) = \dfrac{x+1}{x-2}$.
(From 2025 past paper, Q1 part v)

Solution:
Step 1: Identify numerator and denominator

$$\begin{aligned} u &= x+1 &\implies&& u' &= 1 \\ v &= x-2 &\implies&& v' &= 1 \end{aligned}$$

Step 2: Apply quotient rule
$\left(\dfrac{u}{v}\right)' = \dfrac{vu' - uv'}{v^2}$

$$\begin{aligned} \frac{d}{dx}\left(\frac{x+1}{x-2}\right) &= \frac{(x-2)(1) - (x+1)(1)}{(x-2)^2} \\ &= \frac{x-2-x-1}{(x-2)^2} \\ &= \frac{-3}{(x-2)^2} \end{aligned}$$

Answer: $\boxed{f'(x) = \dfrac{-3}{(x-2)^2}}$

Observation: The derivative is always negative (since $(x-2)^2 > 0$), meaning this function is always decreasing where it's defined!

Example 20: Quotient Rule with Trigonometric Functions

Problem: Find the derivative of $g(x) = \dfrac{\sin x}{1+\cos x}$.

Solution:

Step 1: Identify parts

$$u = \sin x \implies u' = \cos x$$
$$v = 1 + \cos x \implies v' = -\sin x$$

Step 2: Apply quotient rule

$$\begin{aligned} g'(x) &= \frac{(1+\cos x)(\cos x) - (\sin x)(-\sin x)}{(1+\cos x)^2} \\ &= \frac{\cos x + \cos^2 x + \sin^2 x}{(1+\cos x)^2} \end{aligned}$$

Step 3: Simplify using $\sin^2 x + \cos^2 x = 1$

$$\begin{aligned} g'(x) &= \frac{\cos x + 1}{(1+\cos x)^2} \\ &= \frac{1+\cos x}{(1+\cos x)^2} \\ &= \frac{1}{1+\cos x} \end{aligned}$$

Answer: $\boxed{g'(x) = \dfrac{1}{1+\cos x}}$

Tip: Always look for simplification opportunities using trig identities!

Example 21: Quotient Rule - Another Example

Problem: If f and g are both differentiable, find $\dfrac{d}{dx}\left(\dfrac{f(x)}{g(x)}\right)$.

(From 2024 past paper, Q1 part ix - statement of quotient rule)

Solution:

This is asking us to state the quotient rule.

Answer:

$$\boxed{\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{g(x)\cdot f'(x) - f(x)\cdot g'(x)}{[g(x)]^2}}$$

provided $g(x) \neq 0$.

In words:

The derivative of a quotient equals:

$$\frac{(\text{bottom})(\text{derivative of top}) - (\text{top})(\text{derivative of bottom})}{(\text{bottom})^2}$$

Example 22: Combining Product and Quotient Rules

Problem: Find $\dfrac{dy}{dx}$ if $y = \dfrac{x^2+1}{x}\cdot \sin x$.

Solution:

This involves both a quotient and a product.

Method 1: Simplify first, then differentiate
Rewrite:
$$y = \left(\frac{x^2+1}{x}\right)\sin x = \left(x + \frac{1}{x}\right)\sin x = x\sin x + \frac{\sin x}{x}$$
Now differentiate each term using product rule:
For $x \sin x$:
$$\frac{d}{dx}(x\sin x) = x\cos x + \sin x$$
For $\dfrac{\sin x}{x}$ (quotient rule):
$$\frac{d}{dx}\left(\frac{\sin x}{x}\right) = \frac{x\cos x - \sin x}{x^2}$$
Therefore:
$$\frac{dy}{dx} = x\cos x + \sin x + \frac{x\cos x - \sin x}{x^2}$$
Method 2: Use product rule with $u = \dfrac{x^2+1}{x}$ and $v = \sin x$
First find u' using quotient rule:
$$u' = \frac{x(2x) - (x^2+1)(1)}{x^2} = \frac{2x^2 - x^2 - 1}{x^2} = \frac{x^2-1}{x^2}$$
Then apply product rule:
$$\begin{aligned}\frac{dy}{dx} &= u \cdot v' + v \cdot u' \\ &= \frac{x^2+1}{x} \cdot \cos x + \sin x \cdot \frac{x^2-1}{x^2}\end{aligned}$$
Answer (simplified): $\boxed{\dfrac{dy}{dx} = x\cos x + \sin x + \dfrac{x\cos x - \sin x}{x^2}}$

⚠ **Note:** Both methods give the same answer (verify by simplifying!). Choose the method that seems easier for the given problem.

Example 23: Higher Derivatives

Problem: Find $\dfrac{d^2x}{dt^2}$ if $x = \sin t$.
(From 2025 past paper, Q1 part x)

Solution:
The notation $\dfrac{d^2x}{dt^2}$ means the second derivative (derivative of the derivative).
Step 1: Find the first derivative
$$\frac{dx}{dt} = \cos t$$
Step 2: Find the second derivative (derivative of $\cos t$)
$$\frac{d^2x}{dt^2} = \frac{d}{dt}(\cos t) = -\sin t$$

Answer: $\boxed{\dfrac{d^2x}{dt^2} = -\sin t}$

Physical Interpretation: If $x(t)$ is position, then $\dfrac{dx}{dt}$ is velocity and $\dfrac{d^2x}{dt^2}$ is acceleration!

Section 2.4 Summary

Product Rule:

$$\boxed{(fg)' = fg' + gf'}$$

Quotient Rule:

$$\boxed{\left(\frac{f}{g}\right)' = \frac{gf' - fg'}{g^2}}$$

When to Use Which Rule:

Situation	Rule	Example
Two functions multiplied	Product Rule	$x^2 \sin x$
Function divided by function	Quotient Rule	$\dfrac{x+1}{x-2}$
Can simplify first	Do it!	$\dfrac{x^2}{x} = x$
Second derivative	Differentiate twice	$f''(x)$

Strategy Tips:

1. Always check if you can simplify BEFORE applying product/quotient rule
2. For quotient rule: write it as $\dfrac{\text{lo} \cdot \text{dee hi} - \text{hi} \cdot \text{dee lo}}{\text{lo}^2}$
3. For product rule: $(uv)' = u \cdot v' + v \cdot u'$ (order doesn't matter for addition)
4. Always simplify your final answer
5. Use trig identities when applicable

Common Mistakes:

- $(fg)' = f'g'$ WRONG! Use product rule!
- $\left(\dfrac{f}{g}\right)' = \dfrac{f'}{g'}$ WRONG! Use quotient rule!
- Getting the sign wrong in quotient rule (it's $gf' - fg'$, not $fg' - gf'$)
- Forgetting to square the denominator in quotient rule
- Not simplifying algebraically before differentiating

Quick Check:

Can you state these rules from memory?

- Product rule: $(uv)' = ?$
- Quotient rule: $\left(\dfrac{u}{v}\right)' = ?$

If not, practice until you can!

Chain Rule

The chain rule is one of the most powerful and frequently used differentiation techniques. It allows us to differentiate composite functions - functions within functions.

Theorem 2.8: The Chain Rule

If g is differentiable at x and f is differentiable at $g(x)$, then the composite function $F(x) = f(g(x))$ is differentiable at x, and:

$$\boxed{F'(x) = f'(g(x)) \cdot g'(x)}$$

Alternative Notations:
If $y = f(u)$ and $u = g(x)$, then:

$$\boxed{\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}}$$

or

$$\boxed{\frac{d}{dx}[f(g(x))] = f'(g(x)) \cdot g'(x)}$$

In Words:
The derivative of a composite function is:

- The derivative of the outer function (evaluated at the inner function)
- Times
- The derivative of the inner function

Memory Aid:
"Derivative of outside times derivative of inside"

Example 24: Basic Chain Rule Application

Problem: Find the derivative of $f(x) = (x^2 + 1)^3$.

Solution:
This is a composite function: the outer function is u^3 and the inner function is $u = x^2 + 1$.
Step 1: Identify outer and inner functions

$$\text{Inner: } u = g(x) = x^2 + 1$$
$$\text{Outer: } y = f(u) = u^3$$

Step 2: Find derivatives of each part

$$\frac{du}{dx} = 2x$$
$$\frac{dy}{du} = 3u^2$$

Step 3: Apply chain rule
$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$

$$\begin{aligned}\frac{dy}{dx} &= 3u^2 \cdot 2x \\ &= 3(x^2+1)^2 \cdot 2x \\ &= 6x(x^2+1)^2\end{aligned}$$

Answer: $\boxed{f'(x) = 6x(x^2+1)^2}$

Key Steps:

1. Differentiate outer function (keep inside unchanged)
2. Multiply by derivative of inside

Example 25: Chain Rule with Trigonometric Functions

Problem: Find the derivative of $y = \sin(3x^2)$.

Solution:

Step 1: Identify the composition

$$\begin{aligned}\text{Inner: } & u = 3x^2 \\ \text{Outer: } & y = \sin u\end{aligned}$$

Step 2: Find derivatives

$$\begin{aligned}\frac{du}{dx} &= 6x \\ \frac{dy}{du} &= \cos u\end{aligned}$$

Step 3: Apply chain rule

$$\begin{aligned}\frac{dy}{dx} &= \cos u \cdot 6x \\ &= \cos(3x^2) \cdot 6x \\ &= 6x\cos(3x^2)\end{aligned}$$

Answer: $\boxed{\frac{dy}{dx} = 6x\cos(3x^2)}$

Check: The derivative of sin(something) is cos(something) times the derivative of "something".

Example 26: Chain Rule with Exponentials

Problem: Find $\frac{dy}{dx}$ if $y = e^{5x}\sin x$.
(Related to 2025 past paper, Q1 part iv)

Solution:
This requires both the product rule AND the chain rule!
Step 1: Recognize it's a product: $u \cdot v$ where $u = e^{5x}$ and $v = \sin x$

Step 2: Find u' using chain rule
For $u = e^{5x}$:

$$\begin{aligned} u' &= e^{5x} \cdot \frac{d}{dx}(5x) \\ &= e^{5x} \cdot 5 \\ &= 5e^{5x} \end{aligned}$$

Step 3: Find v'

$$v' = \cos x$$

Step 4: Apply product rule
$(uv)' = uv' + vu'$

$$\begin{aligned} \frac{dy}{dx} &= e^{5x} \cdot \cos x + \sin x \cdot 5e^{5x} \\ &= e^{5x} \cos x + 5e^{5x} \sin x \\ &= e^{5x}(\cos x + 5 \sin x) \end{aligned}$$

Answer: $\boxed{\frac{dy}{dx} = e^{5x}(\cos x + 5 \sin x)}$

⚠ **Note:** We factored out e^{5x} for a cleaner form. This is from past paper 2025, Q1(iv)!

Example 27: Multiple Chain Rule Applications

Problem: Find the derivative of:
(a) $f(x) = \sqrt{x^2 + 1}$
(b) $g(x) = (2x + 3)^5$
(c) $h(x) = \cos(x^3)$

Solution:
(a) $f(x) = \sqrt{x^2 + 1} = (x^2 + 1)^{1/2}$
Outer: $u^{1/2}$, Inner: $u = x^2 + 1$

$$\begin{aligned} f'(x) &= \frac{1}{2}(x^2 + 1)^{-1/2} \cdot 2x \\ &= \frac{x}{(x^2 + 1)^{1/2}} \\ &= \frac{x}{\sqrt{x^2 + 1}} \end{aligned}$$

Answer: $\boxed{f'(x) = \frac{x}{\sqrt{x^2 + 1}}}$

(b) $g(x) = (2x + 3)^5$
Outer: u^5, Inner: $u = 2x + 3$

$$\begin{aligned} g'(x) &= 5(2x + 3)^4 \cdot 2 \\ &= 10(2x + 3)^4 \end{aligned}$$

Answer: $\boxed{g'(x) = 10(2x+3)^4}$

(c) $h(x) = \cos(x^3)$
Outer: $\cos u$, Inner: $u = x^3$

$$\begin{aligned} h'(x) &= -\sin(x^3) \cdot 3x^2 \\ &= -3x^2 \sin(x^3) \end{aligned}$$

Answer: $\boxed{h'(x) = -3x^2 \sin(x^3)}$

Example 28: Nested Chain Rule

Problem: Find the derivative of $y = \sin^2(3x) = [\sin(3x)]^2$.

Solution:
This requires the chain rule TWICE (composition of three functions)!
Think of it as: $y = u^2$ where $u = \sin v$ where $v = 3x$.
Step 1: Outermost layer: $y = u^2$ where $u = \sin(3x)$

$$\frac{dy}{du} = 2u = 2\sin(3x)$$

Step 2: Middle layer: $u = \sin(3x)$
Using chain rule again:

$$\frac{du}{dx} = \cos(3x) \cdot 3 = 3\cos(3x)$$

Step 3: Combine using chain rule

$$\begin{aligned} \frac{dy}{dx} &= \frac{dy}{du} \cdot \frac{du}{dx} \\ &= 2\sin(3x) \cdot 3\cos(3x) \\ &= 6\sin(3x)\cos(3x) \end{aligned}$$

Step 4: Optional simplification using double angle formula
Recall: $2\sin\theta\cos\theta = \sin(2\theta)$

$$\begin{aligned} \frac{dy}{dx} &= 6\sin(3x)\cos(3x) \\ &= 3 \cdot 2\sin(3x)\cos(3x) \\ &= 3\sin(6x) \end{aligned}$$

Answer: $\boxed{\frac{dy}{dx} = 6\sin(3x)\cos(3x) = 3\sin(6x)}$

Strategy: Work from outside to inside, applying chain rule at each layer!

Definition 2.8: General Power Chain Rule

A special case of the chain rule that's very useful:

If $u = g(x)$ is differentiable, then:

$$\boxed{\frac{d}{dx}[u^n] = nu^{n-1} \cdot \frac{du}{dx}}$$

In words: Bring down the exponent, reduce the exponent by 1, and multiply by the derivative of the inside.

Examples:

- $\frac{d}{dx}[(x^2+1)^3] = 3(x^2+1)^2 \cdot 2x$
- $\frac{d}{dx}[(3x-2)^{10}] = 10(3x-2)^9 \cdot 3$
- $\frac{d}{dx}[\sqrt{x^2+4}] = \frac{1}{2}(x^2+4)^{-1/2} \cdot 2x$

Example 29: Chain Rule Written Form (Past Paper Style)

Problem: Write the chain rule for differentiation.

(From 2021 past paper, Q1 part ix)

Solution:

The chain rule states that if $y = f(u)$ and $u = g(x)$, then:

Answer:

$$\boxed{\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}}$$

Or equivalently, if $F(x) = f(g(x))$, then:

$$\boxed{F'(x) = f'(g(x)) \cdot g'(x)}$$

In words: The derivative of a composite function is the derivative of the outer function (evaluated at the inner function) times the derivative of the inner function.

Section 2.5 Summary: Chain Rule

The Chain Rule:

$$\boxed{\frac{d}{dx}[f(g(x))] = f'(g(x)) \cdot g'(x)}$$

or

$$\boxed{\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}}$$

Step-by-Step Strategy:

1. Identify the outer function and inner function
2. Differentiate the outer function (keep inside unchanged)

3. Multiply by the derivative of the inner function
4. Simplify

Common Patterns:

Function Type	**Example**	**Derivative**
$(u)^n$	$(x^2+1)^3$	$nu^{n-1} \cdot u'$
$\sin(u)$	$\sin(5x)$	$\cos(u) \cdot u'$
$\cos(u)$	$\cos(x^2)$	$-\sin(u) \cdot u'$
e^u	e^{3x}	$e^u \cdot u'$
$\ln(u)$	$\ln(x^2+1)$	$\frac{1}{u} \cdot u'$

Common Mistakes:

- Forgetting to multiply by the derivative of the inside
- Differentiating the inside function incorrectly
- Not identifying the composition correctly
- Confusing chain rule with product rule

Quick Test: Can you find $\frac{d}{dx}[\sin(x^2)]$ in your head?
Answer: $\cos(x^2) \cdot 2x = 2x\cos(x^2)$ ✓

Implicit Differentiation

So far, we've dealt with functions where y is explicitly written in terms of x, like $y = x^2 + 1$. However, many relationships between variables are given implicitly, such as $x^2 + y^2 = 25$ (a circle). Implicit differentiation allows us to find $\frac{dy}{dx}$ without solving for y first.

Definition 2.9: Implicit vs. Explicit Functions

Explicit Function: y is written explicitly in terms of x.

Examples: $y = x^2 + 3$, $y = \sin x$, $y = \frac{x+1}{x-2}$

Implicit Function: The relationship between x and y is given by an equation.

Examples: $x^2 + y^2 = 25$, $xy = 1$, $x^3 + y^3 = 6xy$

Why Implicit Differentiation?

- Sometimes solving for y explicitly is difficult or impossible
- Even when possible, it may be easier to differentiate implicitly
- Many important curves (circles, ellipses, etc.) are naturally described implicitly

Definition 2.10: Method of Implicit Differentiation

To find $\frac{dy}{dx}$ when y is defined implicitly:

Step 1: Differentiate both sides of the equation with respect to x

Step 2: When differentiating terms with y, use the chain rule:

$$\frac{d}{dx}[f(y)] = f'(y) \cdot \frac{dy}{dx}$$

Step 3: Collect all terms with $\frac{dy}{dx}$ on one side

Step 4: Factor out $\frac{dy}{dx}$ and solve for it

Key Insight:

When we differentiate y with respect to x, we get $\frac{dy}{dx}$ (by chain rule, treating y as a function of x).

Examples:

- $\frac{d}{dx}(y) = \frac{dy}{dx}$
- $\frac{d}{dx}(y^2) = 2y \cdot \frac{dy}{dx}$
- $\frac{d}{dx}(y^3) = 3y^2 \cdot \frac{dy}{dx}$
- $\frac{d}{dx}(\sin y) = \cos y \cdot \frac{dy}{dx}$

Example 30: Basic Implicit Differentiation

Problem: Find $\frac{dy}{dx}$ if $x^2 + y^2 = 25$.

Solution:
This is the equation of a circle with radius 5 centered at the origin.
Step 1: Differentiate both sides with respect to x
Left side:

$$\begin{aligned}\frac{d}{dx}(x^2 + y^2) &= \frac{d}{dx}(x^2) + \frac{d}{dx}(y^2)\\ &= 2x + 2y\frac{dy}{dx}\end{aligned}$$

Right side:

$$\frac{d}{dx}(25) = 0$$

Step 2: Set them equal

$$2x + 2y\frac{dy}{dx} = 0$$

Step 3: Solve for $\frac{dy}{dx}$

$$\begin{aligned}2y\frac{dy}{dx} &= -2x\\ \frac{dy}{dx} &= -\frac{2x}{2y}\\ \frac{dy}{dx} &= -\frac{x}{y}\end{aligned}$$

Answer: $\boxed{\frac{dy}{dx} = -\frac{x}{y}}$

Geometric Interpretation: At any point (x_0, y_0) on the circle, the slope of the tangent line is $-\frac{x_0}{y_0}$.

Check: At $(3, 4)$: slope $= -\frac{3}{4}$. At $(0, 5)$: slope $= 0$ (horizontal tangent).

Example 31: Implicit Differentiation - Finding Tangent Line

Problem: Find the tangent to the curve $x^2y^2 = 9$ at the point $(-1, 3)$.
(From 2021 past paper, Q3 part b)

Solution:
Step 1: Find $\frac{dy}{dx}$ using implicit differentiation
Differentiate both sides:

$$\frac{d}{dx}(x^2y^2) = \frac{d}{dx}(9)$$

Left side (use product rule):

$$\begin{aligned}\frac{d}{dx}(x^2y^2) &= x^2 \cdot \frac{d}{dx}(y^2) + y^2 \cdot \frac{d}{dx}(x^2)\\ &= x^2 \cdot 2y\frac{dy}{dx} + y^2 \cdot 2x\\ &= 2x^2y\frac{dy}{dx} + 2xy^2\end{aligned}$$

Right side:

$$\frac{d}{dx}(9) = 0$$

Step 2: Solve for $\frac{dy}{dx}$

$$\begin{aligned}2x^2y\frac{dy}{dx} + 2xy^2 &= 0\\ 2x^2y\frac{dy}{dx} &= -2xy^2\\ \frac{dy}{dx} &= -\frac{2xy^2}{2x^2y}\\ \frac{dy}{dx} &= -\frac{y}{x}\end{aligned}$$

Step 3: Find slope at $(-1, 3)$

$$\left.\frac{dy}{dx}\right|_{(-1,3)} = -\frac{3}{-1} = 3$$

Step 4: Write equation of tangent line
Using point-slope form with point $(-1, 3)$ and slope $m = 3$:

$$\begin{aligned}y - 3 &= 3(x - (-1))\\ y - 3 &= 3(x + 1)\\ y - 3 &= 3x + 3\\ y &= 3x + 6\end{aligned}$$

Answer: $\boxed{y = 3x + 6}$
Verification: Does $(-1, 3)$ lie on the line? $y = 3(-1) + 6 = 3$ ✓

Example 32: Implicit Differentiation with Trigonometric Functions

Problem: Find $\frac{dy}{dx}$ if $\ln y = e^x \sin x$.
(From 2024 past paper, Q4 part b)

Solution:
Step 1: Differentiate both sides with respect to x
Left side (use chain rule):

$$\frac{d}{dx}(\ln y) = \frac{1}{y} \cdot \frac{dy}{dx}$$

Right side (use product rule):

$$\begin{aligned}\frac{d}{dx}(e^x \sin x) &= e^x \cdot \frac{d}{dx}(\sin x) + \sin x \cdot \frac{d}{dx}(e^x)\\ &= e^x \cos x + \sin x \cdot e^x\\ &= e^x(\cos x + \sin x)\end{aligned}$$

Step 2: Set them equal

$$\frac{1}{y}\frac{dy}{dx} = e^x(\cos x + \sin x)$$

Step 3: Solve for $\frac{dy}{dx}$

$$\frac{dy}{dx} = y \cdot e^x(\cos x + \sin x)$$

But we can also write y in terms of x. From the original equation:

$$\ln y = e^x \sin x \implies y = e^{e^x \sin x}$$

Therefore:

$$\frac{dy}{dx} = e^{e^x \sin x} \cdot e^x(\cos x + \sin x)$$

Answer: $\boxed{\frac{dy}{dx} = ye^x(\cos x + \sin x) = e^{e^x \sin x} \cdot e^x(\cos x + \sin x)}$

Either form is acceptable!

Example 33: Finding Second Derivative Implicitly

Problem: Find y'' if $x^2 + y^4 = 4$.

(From 2023 past paper, Q3 part b)

Solution:

Step 1: Find y' (first derivative) using implicit differentiation

Differentiate both sides:

$$\begin{aligned}\frac{d}{dx}(x^2 + y^4) &= \frac{d}{dx}(4)\\ 2x + 4y^3\frac{dy}{dx} &= 0\end{aligned}$$

Solve for $\frac{dy}{dx}$:

$$\begin{aligned}4y^3\frac{dy}{dx} &= -2x\\ \frac{dy}{dx} &= -\frac{2x}{4y^3}\\ \frac{dy}{dx} &= -\frac{x}{2y^3}\end{aligned}$$

Step 2: Find y'' by differentiating y'

We have $y' = -\frac{x}{2y^3}$. Use the quotient rule:

$$y'' = \frac{d}{dx}\left(-\frac{x}{2y^3}\right)$$

Let $u = -x$ and $v = 2y^3$:

$$u' = -1$$
$$v' = 2 \cdot 3y^2 \cdot \frac{dy}{dx} = 6y^2\frac{dy}{dx}$$

Apply quotient rule:

$$\begin{aligned} y'' &= \frac{v \cdot u' - u \cdot v'}{v^2} \\ &= \frac{2y^3 \cdot (-1) - (-x) \cdot 6y^2 \frac{dy}{dx}}{(2y^3)^2} \\ &= \frac{-2y^3 + 6xy^2 \frac{dy}{dx}}{4y^6} \end{aligned}$$

Step 3: Substitute $\dfrac{dy}{dx} = -\dfrac{x}{2y^3}$

$$\begin{aligned} y'' &= \frac{-2y^3 + 6xy^2 \cdot \left(-\frac{x}{2y^3}\right)}{4y^6} \\ &= \frac{-2y^3 - \frac{6x^2y^2}{2y^3}}{4y^6} \\ &= \frac{-2y^3 - \frac{3x^2}{y}}{4y^6} \end{aligned}$$

Multiply numerator and denominator by y:

$$y'' = \frac{-2y^4 - 3x^2}{4y^7}$$

Answer: $\boxed{y'' = \dfrac{-2y^4 - 3x^2}{4y^7}}$

Alternative form: $\boxed{y'' = -\dfrac{2y^4 + 3x^2}{4y^7}}$

⚠ **Note:** Second derivatives from implicit differentiation can get messy! Always simplify carefully.

Example 34: Implicit Differentiation Practice

Problem: Find y'' if $x^2 + y^4 = 16$.
(From 2025 past paper, Q3)

Solution:
Step 1: Find y' (first derivative)

Differentiate both sides:

$$\begin{aligned} 2x + 4y^3y' &= 0 \\ 4y^3y' &= -2x \\ y' &= -\frac{x}{2y^3} \end{aligned}$$

Step 2: Find y'' using quotient rule

$$y'' = \frac{d}{dx}\left(-\frac{x}{2y^3}\right)$$

Numerator: $u = -x$, so $u' = -1$
Denominator: $v = 2y^3$, so $v' = 6y^2y'$

$$\begin{aligned} y'' &= \frac{(2y^3)(-1) - (-x)(6y^2y')}{(2y^3)^2} \\ &= \frac{-2y^3 + 6xy^2y'}{4y^6} \end{aligned}$$

Step 3: Substitute $y' = -\frac{x}{2y^3}$

$$\begin{aligned} y'' &= \frac{-2y^3 + 6xy^2\left(-\frac{x}{2y^3}\right)}{4y^6} \\ &= \frac{-2y^3 - \frac{3x^2}{y}}{4y^6} \end{aligned}$$

Combine fractions in numerator:

$$\begin{aligned} y'' &= \frac{\frac{-2y^4-3x^2}{y}}{4y^6} \\ &= \frac{-2y^4 - 3x^2}{4y^7} \end{aligned}$$

Answer: $\boxed{y'' = -\frac{2y^4 + 3x^2}{4y^7}}$

Example 35: Implicit Differentiation - Complex Example

Problem: Use implicit differentiation to find $\frac{dy}{dx}$ if $x^3 + y^3 = 6xy$.

Solution:
This is called the Folium of Descartes, a famous curve.
Step 1: Differentiate both sides
Left side:

$$\frac{d}{dx}(x^3 + y^3) = 3x^2 + 3y^2\frac{dy}{dx}$$

Right side (use product rule):

$$\frac{d}{dx}(6xy) = 6\left[x \cdot \frac{dy}{dx} + y \cdot 1\right]$$
$$= 6x\frac{dy}{dx} + 6y$$

Step 2: Set them equal

$$3x^2 + 3y^2\frac{dy}{dx} = 6x\frac{dy}{dx} + 6y$$

Step 3: Collect terms with $\frac{dy}{dx}$

$$3y^2\frac{dy}{dx} - 6x\frac{dy}{dx} = 6y - 3x^2$$
$$(3y^2 - 6x)\frac{dy}{dx} = 6y - 3x^2$$

Step 4: Solve for $\frac{dy}{dx}$

$$\frac{dy}{dx} = \frac{6y - 3x^2}{3y^2 - 6x}$$
$$= \frac{3(2y - x^2)}{3(y^2 - 2x)}$$
$$= \frac{2y - x^2}{y^2 - 2x}$$

Answer: $\boxed{\frac{dy}{dx} = \frac{2y - x^2}{y^2 - 2x}}$

⚠ **Note:** The derivative is undefined when $y^2 - 2x = 0$, i.e., when $y^2 = 2x$. These are points where the tangent line is vertical.

Section 2.6 Summary: Implicit Differentiation

Implicit Differentiation Procedure:

1. Differentiate both sides with respect to x
2. When differentiating y terms, multiply by $\frac{dy}{dx}$ (chain rule)
3. Collect all $\frac{dy}{dx}$ terms on one side
4. Factor out $\frac{dy}{dx}$
5. Solve for $\frac{dy}{dx}$

Key Derivatives to Remember:

Expression	**Derivative w.r.t.** x
y	$\frac{dy}{dx}$
y^n	$ny^{n-1}\frac{dy}{dx}$
$\sin y$	$\cos y \cdot \frac{dy}{dx}$
e^y	$e^y \cdot \frac{dy}{dx}$
$\ln y$	$\frac{1}{y} \cdot \frac{dy}{dx}$
xy	$x\frac{dy}{dx} + y$ (product rule)

When to Use Implicit Differentiation:

- Equation involves both x and y mixed together
- Difficult or impossible to solve for y explicitly
- Finding tangent lines to implicitly defined curves
- Finding second derivatives of implicit functions

Common Mistakes:

- Forgetting to multiply by $\frac{dy}{dx}$ when differentiating y terms
- Not using product rule for terms like xy
- Arithmetic errors when collecting and factoring $\frac{dy}{dx}$ terms
- Not checking if the point satisfies the original equation

Practice Tip:
Always verify your answer by checking:

1. Does the point lie on the original curve?
2. Does the derivative make sense geometrically?
3. Did you collect all $\frac{dy}{dx}$ terms?

Linear Approximation and Differentials

Derivatives provide a powerful tool for approximating function values near a known point. This technique, called linear approximation, is fundamental in numerical analysis, physics, and engineering.

Definition 2.11: Linear Approximation (Linearization)

The **linearization** of a function f at $x = a$ is the linear function:

$$\boxed{L(x) = f(a) + f'(a)(x - a)}$$

This is the equation of the tangent line to f at $x = a$.
For x near a, we have the **linear approximation**:

$$\boxed{f(x) \approx L(x) = f(a) + f'(a)(x - a)}$$

Geometric Interpretation:

- The tangent line is the best linear approximation to the curve
- Near the point of tangency, the line and curve are very close
- The farther from a, the worse the approximation
- This is also called the **tangent line approximation**

Why This Works:
For small $\Delta x = x - a$:

$$f(a + \Delta x) \approx f(a) + f'(a)\Delta x$$

The change in f is approximately the derivative times the change in x.

Example 36: Finding Linearization

Problem: Find the linearization of $f(x) = x + \frac{1}{x}$ at the point $x = 1$.
(From 2023 past paper, Q1 part xi)

Solution:
The linearization is $L(x) = f(a) + f'(a)(x - a)$ where $a = 1$.
Step 1: Find $f(1)$

$$f(1) = 1 + \frac{1}{1} = 1 + 1 = 2$$

Step 2: Find $f'(x)$

$$f(x) = x + \frac{1}{x} = x + x^{-1}$$

$$f'(x) = 1 - x^{-2} = 1 - \frac{1}{x^2}$$

Step 3: Find $f'(1)$

$$f'(1) = 1 - \frac{1}{1^2} = 1 - 1 = 0$$

Step 4: Write the linearization

$$\begin{aligned} L(x) &= f(1) + f'(1)(x-1) \\ &= 2 + 0(x-1) \\ &= 2 \end{aligned}$$

Answer: $\boxed{L(x) = 2}$

Interpretation: Since $f'(1) = 0$, the tangent line is horizontal at $x = 1$. This means the function has a local extremum at this point!

Approximation: For x near 1, $f(x) \approx 2$. For example, $f(1.1) \approx 2$ (actual value: $1.1 + \frac{1}{1.1} \approx 2.009$).

Example 37: Using Linear Approximation

Problem: Find the linearization of $f(x) = x + \dfrac{1}{x}$ at $a = 1$.

(From 2024 past paper, Q1 part vi)

Solution:

This is the same as Example 36.

Step 1: Calculate $f(1)$ and $f'(1)$

From before: $f(1) = 2$ and $f'(1) = 0$

Step 2: Write linearization formula

$$L(x) = f(1) + f'(1)(x-1) = 2 + 0(x-1) = 2$$

Answer: $\boxed{L(x) = 2}$

Note: This appears in both 2023 and 2024 past papers! It's a favorite exam question.

Example 38: Approximating Function Values

Problem: Use linear approximation to estimate $\sqrt{4.1}$.

Solution:

We want to approximate $f(x) = \sqrt{x}$ near $x = 4$ (since $\sqrt{4} = 2$ is known).

Step 1: Set up the problem

Let $f(x) = \sqrt{x}$, $a = 4$, and we want to approximate $f(4.1)$.

Step 2: Find $f(4)$

$$f(4) = \sqrt{4} = 2$$

Step 3: Find $f'(x)$ and $f'(4)$

$$\begin{aligned} f(x) &= x^{1/2} \\ f'(x) &= \frac{1}{2}x^{-1/2} = \frac{1}{2\sqrt{x}} \\ f'(4) &= \frac{1}{2\sqrt{4}} = \frac{1}{2(2)} = \frac{1}{4} \end{aligned}$$

Step 4: Apply linear approximation

$$\begin{aligned} f(4.1) &\approx f(4) + f'(4)(4.1-4) \\ &= 2 + \frac{1}{4}(0.1) \\ &= 2 + 0.025 \\ &= 2.025 \end{aligned}$$

Answer: $\boxed{\sqrt{4.1} \approx 2.025}$

Check: Using a calculator, $\sqrt{4.1} = 2.024845...$, so our approximation is very close! Error < 0.001.

Strategy: Choose a to be a point where:

- $f(a)$ is easy to calculate
- a is close to the desired x value

Definition 2.12: Differentials

If $y = f(x)$ is a differentiable function, we define:

Differential of x (independent variable):

$$\boxed{dx = \Delta x}$$

This is an arbitrary increment in x.

Differential of y (dependent variable):

$$\boxed{dy = f'(x)\,dx}$$

Relationship to Linear Approximation:

The differential dy represents the change in y along the tangent line when x changes by dx.

The actual change in y is:

$$\Delta y = f(x + dx) - f(x)$$

For small dx, we have the approximation:

$$\boxed{\Delta y \approx dy = f'(x)\,dx}$$

Geometric Interpretation:

- Δy: actual change in function value
- dy: change predicted by tangent line
- For small dx, $dy \approx \Delta y$

Example 39: Computing Differentials

Problem: Find the differential dy for:

(a) $y = x^3 + 2x$
(b) $y = \sin(2x)$
(c) $y = \dfrac{1}{x^2}$

Solution:
The formula is $dy = f'(x)\,dx$.
(a) $y = x^3 + 2x$
Find $\dfrac{dy}{dx}$:

$$\frac{dy}{dx} = 3x^2 + 2$$

Therefore:

$$dy = (3x^2 + 2)\,dx$$

Answer: $\boxed{dy = (3x^2 + 2)\,dx}$

(b) $y = \sin(2x)$
Find $\dfrac{dy}{dx}$:

$$\frac{dy}{dx} = \cos(2x) \cdot 2 = 2\cos(2x)$$

Therefore:

$$dy = 2\cos(2x)\,dx$$

Answer: $\boxed{dy = 2\cos(2x)\,dx}$

(c) $y = \dfrac{1}{x^2} = x^{-2}$
Find $\dfrac{dy}{dx}$:

$$\frac{dy}{dx} = -2x^{-3} = -\frac{2}{x^3}$$

Therefore:

$$dy = -\frac{2}{x^3}\,dx$$

Answer: $\boxed{dy = -\dfrac{2}{x^3}\,dx}$

Example 40: Estimating Error Using Differentials

Problem: The radius of a sphere is measured to be 10 cm with a possible error of ± 0.1 cm. Use differentials to estimate the maximum error in the calculated volume.

Solution:
The volume of a sphere is $V = \dfrac{4}{3}\pi r^3$.
Step 1: Find the differential dV

$$\frac{dV}{dr} = \frac{4}{3}\pi \cdot 3r^2 = 4\pi r^2$$

Therefore:

$$dV = 4\pi r^2\,dr$$

Step 2: Substitute known values
Given: $r = 10$ cm, $dr = \pm 0.1$ cm

$$\begin{aligned} dV &= 4\pi(10)^2(\pm 0.1) \\ &= 4\pi(100)(\pm 0.1) \\ &= \pm 40\pi \\ &\approx \pm 125.66 \text{ cm}^3 \end{aligned}$$

Answer: The maximum error in volume is approximately $\boxed{\pm 126 \text{ cm}^3}$

Interpretation: A small error in radius (0.1 cm) leads to a much larger error in volume (≈ 126 cm^3) because volume depends on r^3.

Relative Error:

The actual volume is $V = \frac{4}{3}\pi(10)^3 \approx 4188.79$ cm^3.

Relative error: $\frac{126}{4188.79} \approx 0.03 = 3\%$

Section 2.7 Summary

Linearization at $x = a$:

$$\boxed{L(x) = f(a) + f'(a)(x - a)}$$

Linear Approximation:

$$\boxed{f(x) \approx f(a) + f'(a)(x - a) \quad \text{for } x \text{ near } a}$$

Differentials:

$$\boxed{dy = f'(x)\,dx}$$

Change Approximation:

$$\boxed{\Delta y \approx dy = f'(x)\,dx}$$

Applications:

Application	Method
Approximate function values	Use $f(x) \approx f(a) + f'(a)(x - a)$ with a near x
Estimate errors	Use $\Delta y \approx dy = f'(x)\,dx$
Sensitivity analysis	dy shows how sensitive y is to changes in x
Tangent line equation	$y = L(x)$ is the tangent line

Important Notes:

- Linear approximation is only accurate for x close to a
- The farther from a, the larger the error
- dy approximates Δy but they are not equal

- Always specify units when working with physical quantities

Quick Formula Check:

- Linearization includes: $f(a)$, $f'(a)$, and $(x - a)$
- Differential dy includes: $f'(x)$ and dx

Derivatives of Logarithmic and Inverse Functions

In this section, we explore advanced differentiation techniques involving logarithmic differentiation and inverse functions. These methods are essential for handling complex functions and appear frequently in calculus applications.

Definition 2.13: Logarithmic Differentiation

Logarithmic differentiation is a technique used to differentiate functions of the form:

- Products with many factors
- Quotients with complicated numerators/denominators
- Functions with variables in both base and exponent

Procedure:

1. Take the natural logarithm of both sides: $\ln y = \ln[f(x)]$
2. Use logarithm properties to simplify
3. Differentiate both sides implicitly with respect to x
4. Solve for $\frac{dy}{dx}$
5. Substitute back $y = f(x)$ if needed

Key Logarithm Properties:

- $\ln(ab) = \ln a + \ln b$
- $\ln\left(\frac{a}{b}\right) = \ln a - \ln b$
- $\ln(a^n) = n \ln a$

Remember:
When differentiating $\ln y$ with respect to x:

$$\frac{d}{dx}(\ln y) = \frac{1}{y} \cdot \frac{dy}{dx}$$

Example 41: Basic Logarithmic Differentiation

Problem: Use logarithmic differentiation to find the derivative of $y = \frac{x\sqrt{x^2+1}}{(x+1)^3}$.
(From 2022 past paper, Q5 part b)

Solution:
Step 1: Take natural logarithm of both sides

$$\ln y = \ln\left(\frac{x\sqrt{x^2+1}}{(x+1)^3}\right)$$

Step 2: Use logarithm properties to expand

$$\begin{aligned}\ln y &= \ln(x\sqrt{x^2+1}) - \ln(x+1)^3 \\ &= \ln x + \ln\sqrt{x^2+1} - 3\ln(x+1) \\ &= \ln x + \frac{1}{2}\ln(x^2+1) - 3\ln(x+1)\end{aligned}$$

Step 3: Differentiate both sides with respect to x
Left side:

$$\frac{d}{dx}(\ln y) = \frac{1}{y}\frac{dy}{dx}$$

Right side:

$$\begin{aligned}\frac{d}{dx}\left[\ln x + \frac{1}{2}\ln(x^2+1) - 3\ln(x+1)\right] &= \frac{1}{x} + \frac{1}{2}\cdot\frac{2x}{x^2+1} - 3\cdot\frac{1}{x+1} \\ &= \frac{1}{x} + \frac{x}{x^2+1} - \frac{3}{x+1}\end{aligned}$$

Step 4: Set them equal

$$\frac{1}{y}\frac{dy}{dx} = \frac{1}{x} + \frac{x}{x^2+1} - \frac{3}{x+1}$$

Step 5: Solve for $\frac{dy}{dx}$

$$\begin{aligned}\frac{dy}{dx} &= y\left(\frac{1}{x} + \frac{x}{x^2+1} - \frac{3}{x+1}\right) \\ &= \frac{x\sqrt{x^2+1}}{(x+1)^3}\left(\frac{1}{x} + \frac{x}{x^2+1} - \frac{3}{x+1}\right)\end{aligned}$$

Answer: $\boxed{\frac{dy}{dx} = \frac{x\sqrt{x^2+1}}{(x+1)^3}\left(\frac{1}{x} + \frac{x}{x^2+1} - \frac{3}{x+1}\right)}$

Advantage: Logarithmic differentiation converts multiplication/division into addition/subtraction, making complex derivatives manageable!

Example 42: Logarithmic Differentiation with Variable Exponent

Problem: Find $\frac{dy}{dx}$ if $y = x^x$ (where $x > 0$).

Solution:

This function has a variable base AND exponent, so standard rules don't apply. Use logarithmic differentiation.

Step 1: Take natural logarithm

$$\ln y = \ln(x^x) = x\ln x$$

Step 2: Differentiate both sides
Left side:

$$\frac{d}{dx}(\ln y) = \frac{1}{y}\frac{dy}{dx}$$

Right side (use product rule):

$$\begin{aligned}\frac{d}{dx}(x\ln x) &= x\cdot\frac{1}{x} + \ln x\cdot 1\\ &= 1+\ln x\end{aligned}$$

Step 3: Solve for $\frac{dy}{dx}$

$$\begin{aligned}\frac{1}{y}\frac{dy}{dx} &= 1+\ln x\\ \frac{dy}{dx} &= y(1+\ln x)\\ &= x^x(1+\ln x)\end{aligned}$$

Answer: $\boxed{\frac{dy}{dx} = x^x(1+\ln x)}$

Verification: At $x=1$: $\frac{dy}{dx} = 1^1(1+\ln 1) = 1(1+0) = 1$.
This makes sense because $y = x^x$ passes through $(1,1)$ with slope 1.

Definition 2.14: Derivative of Inverse Functions

If f is a one-to-one differentiable function with inverse f^{-1}, and if $f'(f^{-1}(a)) \neq 0$, then:

$$\boxed{(f^{-1})'(a) = \frac{1}{f'(f^{-1}(a))}}$$

Alternative form: If $y = f^{-1}(x)$, then:

$$\boxed{\frac{dy}{dx} = \frac{1}{dx/dy}}$$

In Words:
The derivative of the inverse function at a is the reciprocal of the derivative of the original function at the corresponding point.

Geometric Interpretation:
The graphs of f and f^{-1} are reflections across the line $y = x$. Their tangent lines are also reflections, so their slopes are reciprocals.

Theorem 2.9: Derivatives of Inverse Trigonometric Functions

$$\boxed{\frac{d}{dx}(\arcsin x) = \frac{1}{\sqrt{1-x^2}}}$$

$$\boxed{\frac{d}{dx}(\arccos x) = -\frac{1}{\sqrt{1-x^2}}}$$

$$\boxed{\frac{d}{dx}(\arctan x) = \frac{1}{1+x^2}}$$

Domain Restrictions:

- $\arcsin x$ and $\arccos x$: defined for $|x| < 1$
- $\arctan x$: defined for all real x

Example 43: Derivative of Inverse Function

Problem: Let $f(x) = x^3 + 1$. Find $(f^{-1})'(2)$.
(Related to 2021 past paper, Q1 part xi)

Solution:
We use the formula: $(f^{-1})'(a) = \dfrac{1}{f'(f^{-1}(a))}$

Step 1: Find $f^{-1}(2)$
We need to solve $f(x) = 2$:

$$\begin{aligned} x^3 + 1 &= 2 \\ x^3 &= 1 \\ x &= 1 \end{aligned}$$

So $f^{-1}(2) = 1$.
Step 2: Find $f'(x)$

$$f'(x) = 3x^2$$

Step 3: Evaluate $f'(f^{-1}(2)) = f'(1)$

$$f'(1) = 3(1)^2 = 3$$

Step 4: Apply the formula

$$(f^{-1})'(2) = \frac{1}{f'(1)} = \frac{1}{3}$$

Answer: $\boxed{(f^{-1})'(2) = \dfrac{1}{3}}$

Verification: We can find $f^{-1}(x)$ explicitly:

$$\begin{aligned} y &= x^3 + 1 \\ x^3 &= y - 1 \\ x &= \sqrt[3]{y-1} \end{aligned}$$

So $f^{-1}(x) = \sqrt[3]{x-1}$.
Then:

$$(f^{-1})'(x) = \frac{1}{3}(x-1)^{-2/3} = \frac{1}{3\sqrt[3]{(x-1)^2}}$$

At $x = 2$:

$$(f^{-1})'(2) = \frac{1}{3\sqrt[3]{(2-1)^2}} = \frac{1}{3\sqrt[3]{1}} = \frac{1}{3}$$

Confirmed! ✓

Example 44: Derivative with Inverse Trigonometric Functions

Problem: Find the derivative of:

(a) $y = \arcsin(2x)$
(b) $y = \arctan(x^2)$
(c) $y = x \arcsin x$

Solution:

(a) $y = \arcsin(2x)$
Use chain rule:

$$\begin{aligned}\frac{dy}{dx} &= \frac{1}{\sqrt{1-(2x)^2}} \cdot 2 \\ &= \frac{2}{\sqrt{1-4x^2}}\end{aligned}$$

Answer: $\boxed{\dfrac{dy}{dx} = \dfrac{2}{\sqrt{1-4x^2}}}$

(b) $y = \arctan(x^2)$
Use chain rule:

$$\begin{aligned}\frac{dy}{dx} &= \frac{1}{1+(x^2)^2} \cdot 2x \\ &= \frac{2x}{1+x^4}\end{aligned}$$

Answer: $\boxed{\dfrac{dy}{dx} = \dfrac{2x}{1+x^4}}$

(c) $y = x \arcsin x$
Use product rule:

$$\begin{aligned}\frac{dy}{dx} &= x \cdot \frac{1}{\sqrt{1-x^2}} + \arcsin x \cdot 1 \\ &= \frac{x}{\sqrt{1-x^2}} + \arcsin x\end{aligned}$$

Answer: $\boxed{\dfrac{dy}{dx} = \dfrac{x}{\sqrt{1-x^2}} + \arcsin x}$

Example 45: Complex Logarithmic Differentiation

Problem: Find $\dfrac{dy}{dx}$ if $y = (x^2+1)^x$.

Solution:
The base and exponent both involve x, so use logarithmic differentiation.
Step 1: Take natural logarithm

$$\ln y = \ln[(x^2+1)^x] = x \ln(x^2+1)$$

Step 2: Differentiate both sides

Left side:

$$\frac{d}{dx}(\ln y) = \frac{1}{y}\frac{dy}{dx}$$

Right side (use product rule):

$$\begin{aligned}\frac{d}{dx}[x\ln(x^2+1)] &= x \cdot \frac{2x}{x^2+1} + \ln(x^2+1)\cdot 1 \\ &= \frac{2x^2}{x^2+1} + \ln(x^2+1)\end{aligned}$$

Step 3: Solve for $\dfrac{dy}{dx}$

$$\begin{aligned}\frac{1}{y}\frac{dy}{dx} &= \frac{2x^2}{x^2+1} + \ln(x^2+1) \\ \frac{dy}{dx} &= y\left[\frac{2x^2}{x^2+1} + \ln(x^2+1)\right] \\ &= (x^2+1)^x\left[\frac{2x^2}{x^2+1} + \ln(x^2+1)\right]\end{aligned}$$

Answer: $\boxed{\dfrac{dy}{dx} = (x^2+1)^x\left[\dfrac{2x^2}{x^2+1} + \ln(x^2+1)\right]}$

Section 2.8 Summary

Logarithmic Differentiation Steps:

1. Take ln of both sides
2. Use log properties to expand
3. Differentiate implicitly
4. Solve for $\dfrac{dy}{dx}$
5. Substitute back $y = f(x)$

When to Use Logarithmic Differentiation:

- Products with many factors
- Complicated quotients
- Variable in both base and exponent: $f(x)^{g(x)}$

Inverse Function Derivative:

$$\boxed{(f^{-1})'(a) = \frac{1}{f'(f^{-1}(a))}}$$

Inverse Trig Derivatives:

Function	Derivative
$\arcsin x$	$\dfrac{1}{\sqrt{1-x^2}}$
$\arccos x$	$-\dfrac{1}{\sqrt{1-x^2}}$
$\arctan x$	$\dfrac{1}{1+x^2}$

Common Mistakes:

- Forgetting to multiply by $\frac{dy}{dx}$ when differentiating $\ln y$
- Not using chain rule with inverse trig functions
- Confusing $(f^{-1})'$ with $\frac{1}{f'}$ (they're not the same!)
- Forgetting to substitute back $y = f(x)$ in logarithmic differentiation

Key Formulas to Memorize:

- $\frac{d}{dx}(\ln y) = \frac{1}{y}\frac{dy}{dx}$
- $\ln(a^n) = n \ln a$
- $\ln(ab) = \ln a + \ln b$
- $\ln\left(\frac{a}{b}\right) = \ln a - \ln b$

Practice Exercises

Problem 1: Derivative from First Principles

Use the definition of the derivative (first principles) to find $f'(x)$ for:

(a) $f(x) = 5x - 3$

(b) $f(x) = x^2 + 2x$

(c) $f(x) = \dfrac{1}{x+1}$

Show all steps clearly, including the limit evaluation.

Problem 2: Basic Derivative Rules

Find the derivative of each function:

(a) $f(x) = 4x^5 - 3x^2 + 7x - 2$

(b) $g(x) = \dfrac{3}{x^4} - 5\sqrt{x} + 2$

(c) $h(x) = 2\sin x + 3\cos x - x^3$

(d) $k(x) = e^x + \ln x - 5x^2 + 1$

(e) $p(x) = \dfrac{2x^3 - 4x}{x^2}$ (simplify first, then differentiate)

Problem 3: Even and Odd Function Derivatives

(a) If $f(x) = x^5 + x$ is an odd function, show that $f'(x)$ is an even function.

(b) If $g(x) = x^4 + 2x^2 - 3$ is an even function, show that $g'(x)$ is an odd function.

(c) State a general rule about derivatives of even and odd functions.

Problem 4: Higher Order Derivatives

Find the first and second derivatives:

(a) $f(x) = x^4 - 6x^2 + 4$. Find $f'(x)$ and $f''(x)$.

(b) $x = \sin t$. Find $\dfrac{dx}{dt}$ and $\dfrac{d^2x}{dt^2}$.

(c) $y = 2x^3 - 3x^2 - 12x + 1$. Find y' and y''.

(d) If $s(t) = t^3 - 6t^2 + 9t$ represents position, find the velocity $v(t)$ and acceleration $a(t)$.

Problem 5: Derivative at a Point

(a) Find $f'(2)$ if $f(x) = x^3 - 4x + 1$.

(b) If $g(x) = \sqrt{x^2 + 5}$, find $g'(2)$.

(c) For $h(x) = \dfrac{x}{x^2+1}$, evaluate $h'(1)$.

(d) If $f(t) = e^t \sin bt$ where b is a constant, find $f'(0)$.

Problem 6: Product Rule Applications

Use the product rule to find the derivative:

(a) $f(x) = (x^2 + 1)(x^3 - 2x)$
(b) $g(x) = x^3 \sin x$
(c) $h(x) = e^x \cos x$
(d) $k(x) = (2x - 1)(x^2 + 3x - 5)$
(e) $p(x) = \ln x \cdot x^2$

Problem 7: Quotient Rule Applications

Use the quotient rule to find $\dfrac{dy}{dx}$:

(a) $y = \dfrac{x^2 + 1}{x - 2}$
(b) $y = \dfrac{\sin x}{x}$
(c) $y = \dfrac{e^x}{x^2 + 1}$
(d) $y = \dfrac{x + 1}{x^2 - 4}$
(e) $y = \dfrac{\ln x}{x}$

For part (a), also find the value of the derivative at $x = 3$.

Problem 8: Chain Rule Practice

Find the derivative using the chain rule:

(a) $f(x) = (x^2 + 3x - 1)^5$
(b) $g(x) = \sin(4x^2)$
(c) $h(x) = e^{3x^2+2x}$
(d) $k(x) = \sqrt{5x^2 - 3x + 1}$
(e) $p(x) = \cos^3(2x)$ [Hint: this is $[\cos(2x)]^3$]
(f) $q(x) = \ln(x^2 + 2x + 5)$

Problem 9: Combined Rules

Find $\dfrac{dy}{dx}$ for the following (may require multiple rules):

(a) $y = x^2 e^{3x}$ (product and chain rule)
(b) $y = \dfrac{\sin(2x)}{x^2}$ (quotient and chain rule)
(c) $y = (x^2 + 1)^3 (x - 2)^2$ (product and chain rule)
(d) $y = \dfrac{e^{2x}}{\sqrt{x}}$ (quotient and chain rule)
(e) $y = \sin(x^2)\cos(3x)$ (product and chain rule)

Problem 10: Trigonometric Derivatives

Find the derivative:

(a) $f(x) = \tan(5x)$
(b) $g(x) = \sec^2(x^2)$
(c) $h(x) = x \tan x$
(d) $k(x) = \dfrac{\sin x}{1 + \cos x}$
(e) $p(x) = \sin^2(3x) + \cos^2(3x)$ [Hint: simplify first!]

Problem 11: Basic Implicit Differentiation

Find $\dfrac{dy}{dx}$ using implicit differentiation:

(a) $x^2 + y^2 = 16$
(b) $x^2 + 4y^2 = 36$
(c) $xy = 12$
(d) $x^3 + y^3 = 6xy$ (Folium of Descartes)
(e) $\sin(xy) = x$

Problem 12: Implicit Differentiation - Tangent Lines

(a) Find the equation of the tangent line to the curve $x^2 + y^2 = 25$ at the point $(3, 4)$.
(b) Find the tangent to the curve $x^2y^2 = 9$ at the point $(-1, 3)$.
(c) For the curve $xy + y^2 = 4$, find the slope of the tangent line at $(1, -2)$.
(d) Find the equation of the normal line to $x^2 + xy + y^2 = 7$ at the point $(1, 2)$.

Problem 13: Second Derivatives - Implicit

Find $\dfrac{d^2y}{dx^2}$ (the second derivative) using implicit differentiation:

(a) $x^2 + y^2 = 25$
(b) $x^2 + y^4 = 4$
(c) $x^2 + y^4 = 16$

Show all steps, including finding $\dfrac{dy}{dx}$ first.

Problem 14: Implicit Differentiation - Special Forms

Find $\dfrac{dy}{dx}$:

(a) $\ln y = e^x \sin x$
(b) $e^{xy} = x + y$
(c) $\sqrt{x} + \sqrt{y} = 1$
(d) $\sin(x + y) = y^2 \cos x$
(e) $y = \sin(xy)$

Problem 15: Verifying Solutions

(a) If $f(x) = \ln(1 + \sqrt{1-x})$, verify that:

$$4x(1-x)f''(x) + 2(2-3x)f'(x) + 1 = 0$$

(b) If $y = e^{ax}\sin(bx)$ where a and b are constants, find y' and y'', then show that:

$$y'' - 2ay' + (a^2 + b^2)y = 0$$

Problem 16: Horizontal and Vertical Tangents

(a) Find all points on the curve $y = x^4 - 6x^2 + 4$ where the tangent line is horizontal.

(b) For $y = x^3 - 3x^2 + 2$, find the points where the slope of the tangent line equals 9.

(c) For the curve $x^2 + xy + y^2 = 3$, find points where the tangent line is horizontal.

(d) Determine if the function $f(x) = 2x^3 - 3x^2 - 12x + 1$ has any horizontal tangents. If so, where?

Problem 17: Tangent and Normal Line Equations

(a) Find the equation of the tangent line to $y = x^2$ at $(2, 4)$.

(b) Find the equation of the normal line to $y = x^2 - 3x + 2$ at $x = 1$.

(c) Find both the tangent and normal lines to $y = \sqrt{x}$ at the point $(4, 2)$.

(d) For the curve $y = e^x$, find the tangent line at $x = 0$. Where does this tangent line cross the x-axis?

(e) Find the tangent line to $f(x) = \dfrac{1}{x}$ at $x = 2$. Does this line pass through the origin?

Problem 18: Linear Approximation

(a) Find the linearization of $f(x) = \sqrt{x}$ at $a = 9$, and use it to approximate $\sqrt{9.2}$.

(b) Find the linearization of $f(x) = x + \dfrac{1}{x}$ at $a = 1$.

(c) Use linear approximation to estimate $\sqrt[3]{8.1}$. [Hint: use $f(x) = \sqrt[3]{x}$ near $a = 8$]

(d) Approximate $\sin(0.1)$ using the linearization of $f(x) = \sin x$ at $a = 0$.

(e) Find the linearization of $f(x) = e^x$ at $a = 0$, and use it to approximate $e^{0.05}$.

Problem 19: Velocity and Acceleration

(a) The position of a particle is given by $s(t) = t^3 - 6t^2 + 9t$ meters.

- Find the velocity $v(t)$ and acceleration $a(t)$
- When is the particle at rest?
- Find the velocity at $t = 2$ seconds

(b) The velocity of a moving object is $v(t) = 4t + 2$ m/s. If the position at $t = 0$ is $s(0) = 5$ m, find the position function $s(t)$.

(c) An object moves with acceleration $a(t) = 6t - 4$ m/s². If $v(0) = 2$ m/s and $s(0) = 1$ m, find the velocity and position functions.

(d) The height of a ball thrown upward is $h(t) = -4.9t^2 + 20t + 2$ meters. Find:

- The velocity at time t
- When the ball reaches its maximum height
- The maximum height

Problem 20: Differentials and Error Estimation

(a) Find the differential dy for:

- $y = x^4 - 2x^2 + 3$
- $y = \sin(3x)$
- $y = \dfrac{x}{x+1}$

(b) The radius of a circle is measured to be 5 cm with a possible error of ± 0.05 cm. Use differentials to estimate the maximum error in:

- The calculated circumference $C = 2\pi r$
- The calculated area $A = \pi r^2$

(c) The side of a cube is measured to be 10 cm with an error of at most 0.02 cm. Use differentials to estimate the maximum error in the calculated volume.

(d) If $f(x) = x^3$, $x = 2$, and $dx = 0.01$, find dy and compare it with $\Delta y = f(2.01) - f(2)$.

Problem 21: Logarithmic Differentiation - Products and Quotients

Use logarithmic differentiation to find $\dfrac{dy}{dx}$:

(a) $y = \dfrac{x\sqrt{x^2+1}}{(x+1)^3}$

(b) $y = \dfrac{(x-1)^2(x+2)^3}{(2x+1)^5}$

(c) $y = \dfrac{x^2\sqrt{x+1}}{(x-2)^3\sqrt{x+3}}$

(d) $y = (x^2+1)^{3/2}(x-1)^{2/3}$

(e) $y = \sqrt{\dfrac{(x+1)(x+2)}{(x-1)(x-2)}}$

Problem 22: Logarithmic Differentiation - Variable Exponents

Find $\dfrac{dy}{dx}$ using logarithmic differentiation:

(a) $y = x^x$ (for $x > 0$)

(b) $y = x^{\sin x}$

(c) $y = (x^2+1)^x$

(d) $y = (\sin x)^x$
(e) $y = x^{x^2}$

Problem 23: Inverse Functions

(a) Let $f(x) = x^3 + 1$. Find $(f^{-1})'(2)$.
(b) If $f(x) = x^5 + x$ and $g = f^{-1}$, find $g'(2)$.
(c) Let $f(x) = 2x + \cos x$. Without finding f^{-1} explicitly, find $(f^{-1})'(1)$.
(d) If $f(x) = x^3 + 2x + 1$ has an inverse function f^{-1}, find $(f^{-1})'(4)$.

Problem 24: Inverse Trigonometric Functions

Find the derivative:
(a) $y = \arcsin(3x)$
(b) $y = \arctan(x^2)$
(c) $y = x \arcsin x$
(d) $y = \dfrac{\arctan x}{x}$
(e) $y = \arcsin(\sqrt{x})$
(f) $y = \arccos\left(\dfrac{1}{x}\right)$
(g) $y = \arctan(e^x)$
(h) $y = \ln(\arctan x)$

Problem 25: Mixed Challenge Problems

(a) Find $\dfrac{dy}{dx}$ if $y = x^{2x}$ (use logarithmic differentiation).
(b) For the curve defined by $e^{x+y} = xy$, find $\dfrac{dy}{dx}$ at the point $(1, 1)$.
(c) If $f(x) = \ln(1 + \sqrt{1-x})$ for $-1 < x < 1$, prove that:

$$4x(1-x)f''(x) + 2(2-3x)f'(x) + 1 = 0$$

(d) Find the derivative of $y = (\sin x)^{\tan x}$.
(e) Use the definition of the derivative to show that:

$$\lim_{x \to 0} \frac{a^x - 1}{x} = \ln a$$

for $a > 0$.
(f) If $y = \sqrt{x + \sqrt{x + \sqrt{x}}}$, find $\dfrac{dy}{dx}$.

Chapter 3

Advanced Differentiation Techniques

Parametric Differentiation

Many curves in mathematics and physics are best described using parametric equations, where both x and y are expressed as functions of a third variable (parameter), usually t. This section develops techniques for finding derivatives of parametric curves.

Definition 3.1: Parametric Equations

A curve in the plane can be described by **parametric equations**:

$$\boxed{x = f(t), \quad y = g(t)}$$

where t is called the **parameter**.

As t varies over an interval, the point $(x(t), y(t))$ traces out a curve in the xy-plane.

Examples of Parametric Curves:

- **Circle:** $x = r\cos t$, $y = r\sin t$ (parameter t is angle)
- **Projectile motion:** $x = v_0 t$, $y = h_0 + v_y t - \frac{1}{2}gt^2$
- **Cycloid:** $x = r(t - \sin t)$, $y = r(1 - \cos t)$

Why Use Parametric Equations?

- Some curves cannot be expressed as $y = f(x)$ (e.g., circles, closed curves)
- Natural for describing motion (time as parameter)
- Multiple y values for same x value allowed
- Easier to describe complex curves

Theorem 3.1: Derivative of Parametric Equations

If $x = f(t)$ and $y = g(t)$ are differentiable functions of t, then the derivative $\dfrac{dy}{dx}$ is given by:

$$\boxed{\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{g'(t)}{f'(t)}}$$

provided that $\frac{dx}{dt} \neq 0$.

Intuition:

Using the chain rule:

$$\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx} = \frac{dy/dt}{dx/dt}$$

We're finding how y changes with respect to x, but computing it through the parameter t.

Important:

- This is NOT $\frac{dy}{dx} = \frac{y}{x}$!
- Don't forget to differentiate with respect to t first
- The denominator is $\frac{dx}{dt}$, not just x

Example 1: Basic Parametric Differentiation

Problem: Find $\frac{dy}{dx}$ if $x = t^2$ and $y = t^3$.

Solution:

Step 1: Find $\frac{dx}{dt}$ and $\frac{dy}{dt}$

$$\frac{dx}{dt} = 2t$$
$$\frac{dy}{dt} = 3t^2$$

Step 2: Apply the parametric derivative formula

$$\begin{aligned}\frac{dy}{dx} &= \frac{dy/dt}{dx/dt} \\ &= \frac{3t^2}{2t} \\ &= \frac{3t}{2}\end{aligned}$$

Answer: $\boxed{\frac{dy}{dx} = \frac{3t}{2}}$

Verification: We can eliminate the parameter:

$$\begin{aligned}&x = t^2 \implies t = \sqrt{x} \\ &y = t^3 = (\sqrt{x})^3 = x^{3/2}\end{aligned}$$

Direct differentiation: $\frac{dy}{dx} = \frac{3}{2}x^{1/2} = \frac{3}{2}\sqrt{x} = \frac{3}{2}\sqrt{t^2} = \frac{3t}{2}$ ✓

Note: The answer is in terms of the parameter t, which is perfectly valid!

Example 2: Parametric Derivative at a Point

Problem: Find $\frac{dy}{dx}$ at $t = 1$ if $x = t^2 + 1$ and $y = 2t^3 - t$.

Solution:

Step 1: Find the derivatives with respect to t

$$\frac{dx}{dt} = 2t$$
$$\frac{dy}{dt} = 6t^2 - 1$$

Step 2: Apply the formula

$$\begin{aligned} \frac{dy}{dx} &= \frac{dy/dt}{dx/dt} \\ &= \frac{6t^2 - 1}{2t} \end{aligned}$$

Step 3: Evaluate at $t = 1$

$$\begin{aligned} \left.\frac{dy}{dx}\right|_{t=1} &= \frac{6(1)^2 - 1}{2(1)} \\ &= \frac{6 - 1}{2} \\ &= \frac{5}{2} \end{aligned}$$

Answer: $\boxed{\left.\frac{dy}{dx}\right|_{t=1} = \frac{5}{2}}$

Geometric Meaning: At $t = 1$, the point is $(2, 1)$ and the slope of the tangent line is $\frac{5}{2}$.

Example 3: Second Derivative of Parametric Equations

Problem: Find $\frac{d^2y}{dx^2}$ if $x = t + \frac{1}{t}$ and $y = t - \frac{1}{t}$.
(From 2024 past paper, Q2 part b)

Solution:

For parametric equations, the second derivative is:

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d/dt\left(\frac{dy}{dx}\right)}{dx/dt}$$

Step 1: Find $\frac{dy}{dx}$

First find the derivatives:

$$\frac{dx}{dt} = 1 - \frac{1}{t^2} = \frac{t^2 - 1}{t^2}$$
$$\frac{dy}{dt} = 1 + \frac{1}{t^2} = \frac{t^2 + 1}{t^2}$$

Therefore:

$$\begin{aligned} \frac{dy}{dx} &= \frac{dy/dt}{dx/dt} \\ &= \frac{(t^2+1)/t^2}{(t^2-1)/t^2} \\ &= \frac{t^2+1}{t^2-1} \end{aligned}$$

Step 2: Find $\dfrac{d}{dt}\left(\dfrac{dy}{dx}\right)$
Use quotient rule:

$$\begin{aligned} \frac{d}{dt}\left(\frac{t^2+1}{t^2-1}\right) &= \frac{(t^2-1)(2t) - (t^2+1)(2t)}{(t^2-1)^2} \\ &= \frac{2t(t^2 - 1 - t^2 - 1)}{(t^2-1)^2} \\ &= \frac{2t(-2)}{(t^2-1)^2} \\ &= \frac{-4t}{(t^2-1)^2} \end{aligned}$$

Step 3: Divide by $\dfrac{dx}{dt}$

$$\begin{aligned} \frac{d^2y}{dx^2} &= \frac{d/dt(dy/dx)}{dx/dt} \\ &= \frac{-4t/(t^2-1)^2}{(t^2-1)/t^2} \\ &= \frac{-4t}{(t^2-1)^2} \cdot \frac{t^2}{t^2-1} \\ &= \frac{-4t^3}{(t^2-1)^3} \end{aligned}$$

Answer: $\boxed{\dfrac{d^2y}{dx^2} = \dfrac{-4t^3}{(t^2-1)^3}}$

⚠ **Note:** Second derivatives in parametric form can get quite complex! Always simplify carefully.

Example 4: Tangent Line to Parametric Curve

Problem: Find the equation of the tangent line to the curve $x = \cos^3 t$, $y = \sin^3 t$ at $t = \frac{\pi}{4}$.

Solution:

Step 1: Find the point on the curve

At $t = \frac{\pi}{4}$:

$$x = \cos^3\left(\frac{\pi}{4}\right) = \left(\frac{1}{\sqrt{2}}\right)^3 = \frac{1}{2\sqrt{2}} = \frac{\sqrt{2}}{4}$$
$$y = \sin^3\left(\frac{\pi}{4}\right) = \left(\frac{1}{\sqrt{2}}\right)^3 = \frac{\sqrt{2}}{4}$$

Point: $\left(\frac{\sqrt{2}}{4}, \frac{\sqrt{2}}{4}\right)$

Step 2: Find the slope

$$\frac{dx}{dt} = 3\cos^2 t \cdot (-\sin t) = -3\cos^2 t \sin t$$
$$\frac{dy}{dt} = 3\sin^2 t \cdot \cos t = 3\sin^2 t \cos t$$

$$\begin{aligned} \frac{dy}{dx} &= \frac{3\sin^2 t \cos t}{-3\cos^2 t \sin t} \\ &= \frac{\sin t}{-\cos t} \\ &= -\tan t \end{aligned}$$

At $t = \frac{\pi}{4}$:

$$\frac{dy}{dx} = -\tan\left(\frac{\pi}{4}\right) = -1$$

Step 3: Write tangent line equation

Using point-slope form:

$$\begin{aligned} y - \frac{\sqrt{2}}{4} &= -1\left(x - \frac{\sqrt{2}}{4}\right) \\ y - \frac{\sqrt{2}}{4} &= -x + \frac{\sqrt{2}}{4} \\ y &= -x + \frac{\sqrt{2}}{2} \end{aligned}$$

Answer: $\boxed{y = -x + \frac{\sqrt{2}}{2}}$

Note: This curve is actually an astroid, which has the Cartesian equation $x^{2/3} + y^{2/3} = 1$.

Definition 3.2: Arc Length of Parametric Curves

The arc length of a parametric curve $x = f(t)$, $y = g(t)$ from $t = a$ to $t = b$ is:

$$\boxed{L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt}$$

Intuition:
The infinitesimal arc length element is:

$$ds = \sqrt{(dx)^2 + (dy)^2} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

We integrate this to get total length.

Relationship to Speed:

If t represents time, then $\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$ is the speed of a particle moving along the curve.

Example 5: Arc Length of Astroid

Problem: Find the length of the astroid $x = \cos^3 t$, $y = \sin^3 t$ for $0 \leq t \leq \pi$.
(From 2022 past paper, Q6 part a - modified)

Solution:

Step 1: Find $\dfrac{dx}{dt}$ and $\dfrac{dy}{dt}$

$$\begin{aligned}\frac{dx}{dt} &= 3\cos^2 t \cdot (-\sin t) = -3\cos^2 t \sin t \\ \frac{dy}{dt} &= 3\sin^2 t \cdot \cos t = 3\sin^2 t \cos t\end{aligned}$$

Step 2: Compute $\left(\dfrac{dx}{dt}\right)^2 + \left(\dfrac{dy}{dt}\right)^2$

$$\begin{aligned}\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 &= 9\cos^4 t \sin^2 t + 9\sin^4 t \cos^2 t \\ &= 9\cos^2 t \sin^2 t(\cos^2 t + \sin^2 t) \\ &= 9\cos^2 t \sin^2 t\end{aligned}$$

Step 3: Set up the arc length integral

$$\begin{aligned}L &= \int_0^\pi \sqrt{9\cos^2 t \sin^2 t}\, dt \\ &= \int_0^\pi 3|\cos t \sin t|\, dt \\ &= 3\int_0^\pi |\cos t \sin t|\, dt\end{aligned}$$

Since $\cos t \sin t \geq 0$ on $[0, \pi/2]$ and $\cos t \sin t \leq 0$ on $[\pi/2, \pi]$:

$$L = 3\int_0^{\pi/2} \cos t \sin t \, dt - 3\int_{\pi/2}^{\pi} \cos t \sin t \, dt$$

Step 4: Evaluate using substitution
Use $\sin t \cos t = \frac{1}{2}\sin(2t)$:

$$\begin{aligned}
L &= 3\int_0^{\pi/2} \frac{1}{2}\sin(2t)\, dt - 3\int_{\pi/2}^{\pi} \frac{1}{2}\sin(2t)\, dt \\
&= \frac{3}{2}\left[-\frac{1}{2}\cos(2t)\right]_0^{\pi/2} - \frac{3}{2}\left[-\frac{1}{2}\cos(2t)\right]_{\pi/2}^{\pi} \\
&= \frac{3}{4}[-\cos\pi + \cos 0] - \frac{3}{4}[-\cos(2\pi) + \cos\pi] \\
&= \frac{3}{4}[1+1] - \frac{3}{4}[-1-1] \\
&= \frac{3}{4}(2) + \frac{3}{4}(2) \\
&= \frac{3}{2} + \frac{3}{2} \\
&= 3
\end{aligned}$$

Answer: The length of the astroid from $t = 0$ to $t = \pi$ is $\boxed{3 \text{ units}}$
Note: The full astroid (one complete loop) from $t = 0$ to $t = 2\pi$ would have length 6 units.

Section 3.1 Summary: Parametric Differentiation

Key Formulas:

First Derivative:

$$\boxed{\frac{dy}{dx} = \frac{dy/dt}{dx/dt}}$$

Second Derivative:

$$\boxed{\frac{d^2y}{dx^2} = \frac{d/dt(dy/dx)}{dx/dt}}$$

Arc Length:

$$\boxed{L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt}$$

Problem-Solving Steps:

1. Find $\frac{dx}{dt}$ and $\frac{dy}{dt}$ separately
2. Use the formula $\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$
3. Simplify the result
4. If finding second derivative, differentiate $\frac{dy}{dx}$ with respect to t, then divide by $\frac{dx}{dt}$

Common Mistakes:

- Writing $\frac{dy}{dx} = \frac{y}{x}$ instead of $\frac{dy/dt}{dx/dt}$
- Forgetting to use chain rule when differentiating parametric equations
- Not simplifying before taking second derivative
- Mixing up numerator and denominator in the parametric derivative formula

Quick Check: Can you state the parametric derivative formula from memory?

Related Rates

Related rates problems involve finding the rate at which one quantity changes with respect to time, given information about the rates of change of related quantities. These problems appear frequently in physics, engineering, and real-world applications.

Definition 3.3: Related Rates

In a **related rates problem**, we are given:

- A relationship (equation) between two or more quantities
- The rate of change of one quantity with respect to time
- We need to find the rate of change of another quantity

General Strategy:

1. Draw a diagram (if applicable)
2. Identify all variables and what's given/unknown
3. Write an equation relating the variables
4. Differentiate both sides with respect to time t (using implicit differentiation and chain rule)
5. Substitute known values
6. Solve for the unknown rate

Key Insight:

When we differentiate with respect to time, every variable that depends on time gets differentiated using the chain rule:

- $\frac{d}{dt}(x) = \frac{dx}{dt}$ (often written as x' or $\dot{x}$)
- $\frac{d}{dt}(x^2) = 2x\frac{dx}{dt}$
- $\frac{d}{dt}(xy) = x\frac{dy}{dt} + y\frac{dx}{dt}$ (product rule)

Important:

- Don't substitute numerical values until AFTER differentiating
- Be careful with units (consistent units throughout)
- Positive/negative signs indicate direction of change

Example 6: Expanding Circle

Problem: The radius of a circle is increasing at a rate of 2 cm/s. How fast is the area increasing when the radius is 5 cm?

Solution:

Step 1: Identify what's given and what we need to find
Given: $\frac{dr}{dt} = 2$ cm/s (radius increasing)
Find: $\frac{dA}{dt}$ when $r = 5$ cm (rate of change of area)
Step 2: Write the relationship between variables
For a circle: $A = \pi r^2$
Step 3: Differentiate both sides with respect to t

$$\begin{aligned}\frac{dA}{dt} &= \frac{d}{dt}(\pi r^2)\\ &= \pi \cdot 2r \cdot \frac{dr}{dt}\\ &= 2\pi r\frac{dr}{dt}\end{aligned}$$

Step 4: Substitute known values
When $r = 5$ cm and $\frac{dr}{dt} = 2$ cm/s:

$$\begin{aligned}\frac{dA}{dt} &= 2\pi(5)(2)\\ &= 20\pi\\ &\approx 62.83\ \text{cm}^2/\text{s}\end{aligned}$$

Answer: The area is increasing at $\boxed{20\pi \approx 62.83\ \text{cm}^2/\text{s}}$ when $r = 5$ cm.

Interpretation: As the radius increases, the area increases even faster (proportional to r). At $r = 10$ cm, the area would be increasing at 40π cm²/s!

Example 7: Ladder Sliding Down a Wall

Problem: A 10-meter ladder is leaning against a wall. The bottom of the ladder is sliding away from the wall at a rate of 1 m/s. How fast is the top of the ladder sliding down the wall when the bottom is 6 meters from the wall?

Solution:
Step 1: Draw a diagram and label variables
Let:

- x = distance from wall to bottom of ladder (horizontal)
- y = height of top of ladder on wall (vertical)
- $L = 10$ m = length of ladder (constant)

Given: $\frac{dx}{dt} = 1$ m/s (bottom moving away from wall)
Find: $\frac{dy}{dt}$ when $x = 6$ m
Step 2: Write equation relating variables
By Pythagorean theorem:

$$x^2 + y^2 = L^2 = 100$$

Step 3: Differentiate with respect to t

$$\frac{d}{dt}(x^2 + y^2) = \frac{d}{dt}(100)$$
$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$$

Simplify:

$$x\frac{dx}{dt} + y\frac{dy}{dt} = 0$$

Step 4: Find y when $x = 6$

$$\begin{aligned} 6^2 + y^2 &= 100 \\ y^2 &= 64 \\ y &= 8 \text{ m} \quad \text{(positive since on wall)} \end{aligned}$$

Step 5: Substitute and solve for $\dfrac{dy}{dt}$

$$\begin{aligned} 6(1) + 8\frac{dy}{dt} &= 0 \\ 8\frac{dy}{dt} &= -6 \\ \frac{dy}{dt} &= -\frac{6}{8} = -\frac{3}{4} \text{ m/s} \end{aligned}$$

Answer: The top of the ladder is sliding down at $\boxed{-\frac{3}{4} = -0.75 \text{ m/s}}$

The negative sign indicates the height is decreasing (sliding down).

Check: Units work out: m/s is correct for velocity.

Example 8: Expanding Sphere

Problem: A spherical balloon is being inflated at a rate of 100 cm³/s. How fast is the radius increasing when the radius is 10 cm?

Solution:

Step 1: Identify variables

Given: $\dfrac{dV}{dt} = 100$ cm³/s (volume increasing)

Find: $\dfrac{dr}{dt}$ when $r = 10$ cm

Step 2: Relate variables

Volume of sphere: $V = \dfrac{4}{3}\pi r^3$

Step 3: Differentiate with respect to t

$$\begin{aligned}\frac{dV}{dt} &= \frac{d}{dt}\left(\frac{4}{3}\pi r^3\right)\\ &= \frac{4}{3}\pi \cdot 3r^2 \cdot \frac{dr}{dt}\\ &= 4\pi r^2 \frac{dr}{dt}\end{aligned}$$

Step 4: Solve for $\frac{dr}{dt}$

$$\frac{dr}{dt} = \frac{1}{4\pi r^2} \cdot \frac{dV}{dt}$$

Step 5: Substitute values
When $r = 10$ cm and $\frac{dV}{dt} = 100$ cm³/s:

$$\begin{aligned}\frac{dr}{dt} &= \frac{100}{4\pi(10)^2}\\ &= \frac{100}{400\pi}\\ &= \frac{1}{4\pi}\\ &\approx 0.0796 \text{ cm/s}\end{aligned}$$

Answer: The radius is increasing at $\boxed{\frac{1}{4\pi} \approx 0.08 \text{ cm/s}}$ when $r = 10$ cm.

Observation: As the balloon gets larger, the same volume increase causes a smaller change in radius because the surface area is larger.

Example 9: Water Tank Problem

Problem: Water is draining from a conical tank at a rate of 2 m³/min. The tank has height 10 m and radius 5 m at the top. How fast is the water level dropping when the water is 6 m deep?

Solution:
Step 1: Set up variables
Let:

- h = height of water (depth)
- r = radius of water surface
- V = volume of water

Given: $\frac{dV}{dt} = -2$ m³/min (negative because draining)
Find: $\frac{dh}{dt}$ when $h = 6$ m
Step 2: Relate r and h
By similar triangles (cone proportions):

$$\frac{r}{h} = \frac{5}{10} = \frac{1}{2} \implies r = \frac{h}{2}$$

Step 3: Write volume formula

Volume of cone: $V = \frac{1}{3}\pi r^2 h$

Substitute $r = \frac{h}{2}$:

$$\begin{aligned} V &= \frac{1}{3}\pi\left(\frac{h}{2}\right)^2 h \\ &= \frac{1}{3}\pi \cdot \frac{h^2}{4} \cdot h \\ &= \frac{\pi h^3}{12} \end{aligned}$$

Step 4: Differentiate with respect to t

$$\begin{aligned} \frac{dV}{dt} &= \frac{\pi}{12} \cdot 3h^2 \cdot \frac{dh}{dt} \\ &= \frac{\pi h^2}{4} \cdot \frac{dh}{dt} \end{aligned}$$

Step 5: Solve for $\frac{dh}{dt}$

$$\frac{dh}{dt} = \frac{4}{\pi h^2} \cdot \frac{dV}{dt}$$

Step 6: Substitute values

When $h = 6$ m and $\frac{dV}{dt} = -2\ \text{m}^3/\text{min}$:

$$\begin{aligned} \frac{dh}{dt} &= \frac{4}{\pi(6)^2} \cdot (-2) \\ &= \frac{-8}{36\pi} \\ &= -\frac{2}{9\pi} \\ &\approx -0.071\ \text{m/min} \end{aligned}$$

Answer: The water level is dropping at $\boxed{\frac{2}{9\pi} \approx 0.071\ \text{m/min}}$ when $h = 6$ m. The negative sign indicates the height is decreasing.

Example 10: Distance Between Moving Objects

Problem: Car A is traveling north at 60 km/h and car B is traveling west at 80 km/h. Both cars are approaching an intersection. At a certain moment, car A is 0.5 km south of the intersection and car B is 0.6 km east of the intersection. How fast is the distance between the cars changing at that moment?

Solution:

Step 1: Set up coordinates and variables

Let:

- x = distance of car B from intersection (east)

- y = distance of car A from intersection (south)
- z = distance between the two cars

Given:

- $\dfrac{dy}{dt} = -60$ km/h (negative: moving north, decreasing y)
- $\dfrac{dx}{dt} = -80$ km/h (negative: moving west, decreasing x)
- At the moment: $y = 0.5$ km, $x = 0.6$ km

Find: $\dfrac{dz}{dt}$

Step 2: Relate variables using Pythagorean theorem

$$z^2 = x^2 + y^2$$

Step 3: Find z at the given moment

$$\begin{aligned} z^2 &= (0.6)^2 + (0.5)^2 \\ &= 0.36 + 0.25 \\ &= 0.61 \\ z &= \sqrt{0.61} \approx 0.781 \text{ km} \end{aligned}$$

Step 4: Differentiate with respect to t

$$2z\frac{dz}{dt} = 2x\frac{dx}{dt} + 2y\frac{dy}{dt}$$

Simplify:

$$z\frac{dz}{dt} = x\frac{dx}{dt} + y\frac{dy}{dt}$$

Step 5: Substitute values

$$\begin{aligned} \sqrt{0.61} \cdot \frac{dz}{dt} &= 0.6(-80) + 0.5(-60) \\ 0.781 \cdot \frac{dz}{dt} &= -48 - 30 \\ 0.781 \cdot \frac{dz}{dt} &= -78 \\ \frac{dz}{dt} &= \frac{-78}{0.781} \\ &\approx -99.87 \text{ km/h} \end{aligned}$$

Answer: The distance between the cars is decreasing at approximately $\boxed{100 \text{ km/h}}$

The negative sign indicates they are getting closer (distance decreasing).

Reality Check: This makes sense! Both cars are heading toward the intersection, so they're approaching each other rapidly.

Section 3.2 Summary: Related Rates

General Problem-Solving Strategy:

1. **Draw and label:** Sketch the situation with variables
2. **Identify:** What's given (rates) and what's unknown
3. **Relate:** Write equation connecting all variables
4. **Differentiate:** Use implicit differentiation with respect to t
5. **Don't substitute early:** Differentiate first, then substitute
6. **Solve:** Find the unknown rate
7. **Check:** Units and reasonableness of answer

Common Relationships:

Situation	Key Formula
Circle	$A = \pi r^2$
Sphere	$V = \frac{4}{3}\pi r^3$, $S = 4\pi r^2$
Cone	$V = \frac{1}{3}\pi r^2 h$
Right Triangle	$a^2 + b^2 = c^2$
Similar Triangles	$\frac{a}{b} = \frac{c}{d}$

Common Mistakes:

- Substituting values before differentiating
- Forgetting negative signs (direction matters!)
- Not using chain rule when differentiating
- Mixing up which rate is given and which is unknown
- Unit inconsistencies
- Not relating variables correctly before differentiating

Sign Convention:

- Positive rate: quantity is increasing
- Negative rate: quantity is decreasing
- Always interpret the sign in context of the problem

Higher Order Derivatives and Applications

Higher order derivatives (second, third, fourth derivatives, etc.) provide important information about the behavior of functions. They are essential in physics, engineering, and understanding the geometry of curves.

Definition 3.4: Higher Order Derivatives

If f is a differentiable function, then its **derivative** f' is also a function. If f' is differentiable, we can take its derivative to get the **second derivative**:

$$\boxed{f''(x) = \frac{d}{dx}[f'(x)] = \frac{d^2 f}{dx^2}}$$

Continuing this process:

Third derivative:

$$f'''(x) = \frac{d^3 f}{dx^3}$$

Fourth derivative:

$$f^{(4)}(x) = \frac{d^4 f}{dx^4}$$

n-th derivative:

$$f^{(n)}(x) = \frac{d^n f}{dx^n}$$

Notation Variations:

For $y = f(x)$:

$$\text{First derivative: } f'(x),\ y',\ \frac{dy}{dx},\ \frac{df}{dx},\ Df$$

$$\text{Second derivative: } f''(x),\ y'',\ \frac{d^2y}{dx^2},\ D^2 f$$

$$\text{Third derivative: } f'''(x),\ y''',\ \frac{d^3y}{dx^3},\ D^3 f$$

$$\text{Higher orders: } f^{(n)}(x),\ y^{(n)},\ \frac{d^ny}{dx^n},\ D^n f$$

Note: After the third derivative, we use the notation $f^{(4)}$, $f^{(5)}$, etc., not f'''' (too many primes!).

Example 11: Computing Higher Derivatives

Problem: Find the first four derivatives of $f(x) = x^5 - 3x^4 + 2x^2 - 5$.

Solution:

First derivative:

$$f'(x) = 5x^4 - 12x^3 + 4x$$

Second derivative:

$$f''(x) = 20x^3 - 36x^2 + 4$$

Third derivative:

$$f'''(x) = 60x^2 - 72x$$

Fourth derivative:

$$f^{(4)}(x) = 120x - 72$$

Fifth derivative:

$$f^{(5)}(x) = 120$$

Sixth derivative and beyond:

$$\begin{aligned} f^{(6)}(x) &= 0 \\ f^{(n)}(x) &= 0 \quad \text{for all } n \geq 6 \end{aligned}$$

Answer:

$$\begin{aligned} f'(x) &= \boxed{5x^4 - 12x^3 + 4x} \\ f''(x) &= \boxed{20x^3 - 36x^2 + 4} \\ f'''(x) &= \boxed{60x^2 - 72x} \\ f^{(4)}(x) &= \boxed{120x - 72} \end{aligned}$$

Pattern: For polynomial of degree n, the n-th derivative is constant, and all higher derivatives are zero.

Example 12: Higher Derivatives of Exponential and Trig Functions

Problem: Find the n-th derivative for:

(a) $f(x) = e^x$
(b) $g(x) = \sin x$
(c) $h(x) = \cos x$

Solution:

(a) For $f(x) = e^x$:

$$\begin{aligned} f'(x) &= e^x \\ f''(x) &= e^x \\ f'''(x) &= e^x \\ &\vdots \\ f^{(n)}(x) &= e^x \end{aligned}$$

Answer: $\boxed{f^{(n)}(x) = e^x \text{ for all } n \geq 1}$

The exponential function is its own derivative!

(b) For $g(x) = \sin x$:

$$\begin{aligned} g'(x) &= \cos x \\ g''(x) &= -\sin x \\ g'''(x) &= -\cos x \\ g^{(4)}(x) &= \sin x \\ g^{(5)}(x) &= \cos x \\ &\vdots \end{aligned}$$

The pattern repeats every 4 derivatives:
Answer:

$$\boxed{g^{(n)}(x) = \begin{cases} \sin x, & n \equiv 0 \pmod 4 \\ \cos x, & n \equiv 1 \pmod 4 \\ -\sin x, & n \equiv 2 \pmod 4 \\ -\cos x, & n \equiv 3 \pmod 4 \end{cases}}$$

(c) For $h(x) = \cos x$:

$$\begin{aligned} h'(x) &= -\sin x \\ h''(x) &= -\cos x \\ h'''(x) &= \sin x \\ h^{(4)}(x) &= \cos x \\ h^{(5)}(x) &= -\sin x \\ &\vdots \end{aligned}$$

Answer: Pattern repeats every 4 derivatives:

$$\boxed{h^{(n)}(x) = \begin{cases} \cos x, & n \equiv 0 \pmod 4 \\ -\sin x, & n \equiv 1 \pmod 4 \\ -\cos x, & n \equiv 2 \pmod 4 \\ \sin x, & n \equiv 3 \pmod 4 \end{cases}}$$

Definition 3.5: Physical Interpretation of Higher Derivatives

In physics and mechanics, derivatives with respect to time have special meanings:

Position, Velocity, Acceleration:

If $s(t)$ is the position of an object at time t:

First derivative (velocity):

$$v(t) = s'(t) = \frac{ds}{dt}$$

Second derivative (acceleration):

$$a(t) = v'(t) = s''(t) = \frac{d^2s}{dt^2}$$

Third derivative (jerk):

$$j(t) = a'(t) = s'''(t) = \frac{d^3s}{dt^3}$$

What is Jerk?

Jerk measures the rate of change of acceleration. It's what you feel when:

- An elevator starts or stops suddenly
- A car accelerates or brakes abruptly
- A roller coaster changes acceleration

Question from 2024 Past Paper:

"What is jerk? Why is the jerk during free fall zero?" (Q1 part v)

Answer: During free fall, acceleration is constant ($a = -g$), so $j = \dfrac{da}{dt} = 0$.

Example 13: Position Velocity Acceleration and Jerk

Problem: A particle moves along a line with position function $s(t) = t^4 - 8t^3 + 24t^2 - 32t + 10$ meters, where t is in seconds.

(a) Find the velocity, acceleration, and jerk functions.
(b) Find the velocity and acceleration at $t = 2$ seconds.
(c) When is the jerk zero?

Solution:

(a) Find $v(t)$, $a(t)$, and $j(t)$

Velocity:

$$v(t) = s'(t) = 4t^3 - 24t^2 + 48t - 32$$

Acceleration:

$$a(t) = v'(t) = s''(t) = 12t^2 - 48t + 48$$

Jerk:

$$j(t) = a'(t) = s'''(t) = 24t - 48$$

Answer:

$$v(t) = \boxed{4t^3 - 24t^2 + 48t - 32 \text{ m/s}}$$
$$a(t) = \boxed{12t^2 - 48t + 48 \text{ m/s}^2}$$
$$j(t) = \boxed{24t - 48 \text{ m/s}^3}$$

(b) Evaluate at $t = 2$

$$\begin{aligned} v(2) &= 4(2)^3 - 24(2)^2 + 48(2) - 32 \\ &= 32 - 96 + 96 - 32 \\ &= 0 \text{ m/s} \end{aligned}$$

$$\begin{aligned} a(2) &= 12(2)^2 - 48(2) + 48 \\ &= 48 - 96 + 48 \\ &= 0 \text{ m/s}^2 \end{aligned}$$

Answer: At $t = 2$ seconds: $\boxed{v = 0 \text{ m/s}, a = 0 \text{ m/s}^2}$
The particle is momentarily at rest with zero acceleration.

(c) When is jerk zero?

$$\begin{aligned} j(t) &= 0 \\ 24t - 48 &= 0 \\ t &= 2 \text{ seconds} \end{aligned}$$

Answer: Jerk is zero at $\boxed{t = 2 \text{ s}}$
Interpretation: At $t = 2$ seconds, the particle is at rest, has zero acceleration, and zero jerk. This is a very special point in the motion!

Definition 3.6: Concavity and Second Derivative

The second derivative provides information about the **concavity** of a function:

Concave Up: $f''(x) > 0$

- Graph curves upward (like $\cup$)
- Function is "holding water"
- Slope is increasing

Concave Down: $f''(x) < 0$

- Graph curves downward (like $\cap$)
- Function is "spilling water"
- Slope is decreasing

Inflection Point: A point where concavity changes

- Occurs where $f''(x) = 0$ or $f''(x)$ is undefined
- Must verify that f'' changes sign

Visual Guide:

- If $f''(x) > 0$: tangent lines lie below the curve
- If $f''(x) < 0$: tangent lines lie above the curve

Example 14: Finding Inflection Points

Problem: For $f(x) = 2x^3 - 3x^2 - 12x$:
(a) Find the inflection point(s).
(b) Determine where the function is concave up and concave down.
(From 2024 past paper, Q2 part a)

Solution:
(a) Find inflection points
Step 1: Find the second derivative

$$\begin{aligned} f'(x) &= 6x^2 - 6x - 12 \\ f''(x) &= 12x - 6 \end{aligned}$$

Step 2: Set $f''(x) = 0$

$$\begin{aligned} 12x - 6 &= 0 \\ 12x &= 6 \\ x &= \frac{1}{2} \end{aligned}$$

Step 3: Verify that concavity changes
Test points around $x = \frac{1}{2}$:
For $x = 0$ (left of $\frac{1}{2}$): $f''(0) = -6 < 0$ (concave down)
For $x = 1$ (right of $\frac{1}{2}$): $f''(1) = 12 - 6 = 6 > 0$ (concave up)
Concavity changes! ✓
Step 4: Find the y-coordinate

$$\begin{aligned} f\left(\frac{1}{2}\right) &= 2\left(\frac{1}{2}\right)^3 - 3\left(\frac{1}{2}\right)^2 - 12\left(\frac{1}{2}\right) \\ &= 2 \cdot \frac{1}{8} - 3 \cdot \frac{1}{4} - 6 \\ &= \frac{1}{4} - \frac{3}{4} - 6 \\ &= -\frac{1}{2} - 6 \\ &= -\frac{13}{2} \end{aligned}$$

Answer (a): Inflection point at $\boxed{\left(\frac{1}{2}, -\frac{13}{2}\right)}$

(b) Intervals of concavity
From our analysis:
Concave down: $f''(x) < 0$ when $12x - 6 < 0$, i.e., $x < \frac{1}{2}$
Interval: $\boxed{\left(-\infty, \frac{1}{2}\right)}$
Concave up: $f''(x) > 0$ when $12x - 6 > 0$, i.e., $x > \frac{1}{2}$
Interval: $\boxed{\left(\frac{1}{2}, \infty\right)}$

Theorem 3.2: Second Derivative Test for Extrema

Let f be twice differentiable and suppose $f'(c) = 0$.

If $f''(c) > 0$: f has a **local minimum** at $x = c$

If $f''(c) < 0$: f has a **local maximum** at $x = c$

If $f''(c) = 0$: Test is **inconclusive** (could be min, max, or neither)

Intuition:

- $f''(c) > 0$ means concave up at critical point → local minimum
- $f''(c) < 0$ means concave down at critical point → local maximum

Requirement: This test only works at critical points where $f'(c) = 0$!

Example 15: Using Second Derivative Test

Problem: Find and classify all local extrema of $f(x) = x^4 - 4x^3 + 6$.

Solution:

Step 1: Find critical points

$$f'(x) = 4x^3 - 12x^2 = 4x^2(x - 3)$$

Set $f'(x) = 0$:

$$4x^2(x - 3) = 0 \implies x = 0 \text{ or } x = 3$$

Critical points: $x = 0$ and $x = 3$

Step 2: Find second derivative

$$f''(x) = 12x^2 - 24x = 12x(x - 2)$$

Step 3: Apply second derivative test

At $x = 0$:

$$f''(0) = 12(0)(0 - 2) = 0$$

Test is inconclusive! Need first derivative test or further analysis.

At $x = 3$:

$$f''(3) = 12(3)(3 - 2) = 12(3)(1) = 36 > 0$$

Since $f''(3) > 0$, there is a **local minimum** at $x = 3$.

Step 4: Find function values

$$f(0) = 0 - 0 + 6 = 6$$
$$f(3) = 81 - 108 + 6 = -21$$

Step 5: Analyze $x = 0$ using first derivative test

Check sign of $f'(x) = 4x^2(x - 3)$ around $x = 0$:

- For $x = -1$: $f'(-1) = 4(1)(-4) = -16 < 0$ (decreasing)
- For $x = 1$: $f'(1) = 4(1)(-2) = -8 < 0$ (decreasing)

No sign change → $x = 0$ is **neither a max nor min** (inflection point).

Answer:

- Local minimum at $\boxed{(3, -21)}$
- No local maximum
- $x = 0$ is not an extremum

Section 3.3 Summary

Higher Order Derivatives:

$$f''(x) = \frac{d^2 f}{dx^2} = \frac{d}{dx}[f'(x)]$$
$$f'''(x) = \frac{d^3 f}{dx^3}$$
$$f^{(n)}(x) = \frac{d^n f}{dx^n}$$

Physical Interpretations:

Derivative	Name	Meaning
$s'(t)$	Velocity	Rate of change of position
$s''(t)$	Acceleration	Rate of change of velocity
$s'''(t)$	Jerk	Rate of change of acceleration

Concavity:

- $f''(x) > 0 \to$ Concave up ($\cup$)
- $f''(x) < 0 \to$ Concave down ($\cap$)
- $f''(x) = 0$ and changes sign $\to$ Inflection point

Second Derivative Test:

- $f'(c) = 0$ and $f''(c) > 0 \to$ Local minimum
- $f'(c) = 0$ and $f''(c) < 0 \to$ Local maximum
- $f'(c) = 0$ and $f''(c) = 0 \to$ Inconclusive

Common Mistakes:

- Confusing inflection points with extrema
- Forgetting to verify sign change for inflection points
- Using second derivative test when $f'(c) \neq 0$
- Not checking that f'' actually exists at the point

Newton-Raphson Method

The Newton-Raphson method (also called Newton's method) is a powerful numerical technique for finding approximate solutions to equations. It uses derivatives to iteratively improve approximations to roots of equations.

Definition 3.7: Newton-Raphson Method

To find a root of the equation $f(x) = 0$, start with an initial guess x_0 and generate a sequence of approximations using:

$$\boxed{x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}}$$

for $n = 0, 1, 2, 3, \ldots$

Geometric Interpretation:

1. Start at point $(x_n, f(x_n))$ on the curve
2. Draw the tangent line at this point
3. Find where the tangent line crosses the x-axis
4. This x-intercept becomes x_{n+1} (next approximation)
5. Repeat the process

The Formula Derivation:

The tangent line at $(x_n, f(x_n))$ has equation:

$$y - f(x_n) = f'(x_n)(x - x_n)$$

Setting $y = 0$ (to find x-intercept):

$$\begin{aligned} -f(x_n) &= f'(x_n)(x - x_n) \\ x - x_n &= -\frac{f(x_n)}{f'(x_n)} \\ x &= x_n - \frac{f(x_n)}{f'(x_n)} \end{aligned}$$

This gives us x_{n+1}.

Requirements:

- $f'(x_n) \neq 0$ (tangent line not horizontal)
- Good initial guess x_0 (close to actual root)
- Function should be "well-behaved" near the root

Example 16: Newton-Raphson Method - Basic Application

Problem: Use Newton-Raphson method to approximate $\sqrt{2}$ (i.e., solve $x^2 - 2 = 0$) starting with $x_0 = 1$. Find three iterations.

Solution:

Step 1: Set up the function and its derivative

$$\begin{aligned} f(x) &= x^2 - 2 \\ f'(x) &= 2x \end{aligned}$$

Step 2: Write the iteration formula

$$\begin{aligned} x_{n+1} &= x_n - \frac{f(x_n)}{f'(x_n)} \\ &= x_n - \frac{x_n^2 - 2}{2x_n} \\ &= x_n - \frac{x_n}{2} + \frac{1}{x_n} \\ &= \frac{x_n}{2} + \frac{1}{x_n} \\ &= \frac{x_n^2 + 2}{2x_n} \end{aligned}$$

Step 3: Perform iterations
Iteration 0: $x_0 = 1$ (given)
Iteration 1:

$$x_1 = \frac{x_0^2 + 2}{2x_0} = \frac{1+2}{2(1)} = \frac{3}{2} = 1.5$$

Iteration 2:

$$x_2 = \frac{x_1^2 + 2}{2x_1} = \frac{(1.5)^2 + 2}{2(1.5)} = \frac{2.25 + 2}{3} = \frac{4.25}{3} \approx 1.41667$$

Iteration 3:

$$\begin{aligned} x_3 &= \frac{x_2^2 + 2}{2x_2} = \frac{(1.41667)^2 + 2}{2(1.41667)} \\ &\approx \frac{2.00694 + 2}{2.83334} \approx \frac{4.00694}{2.83334} \approx 1.41421 \end{aligned}$$

Answer:

$$\begin{aligned} x_1 &= \boxed{1.5} \\ x_2 &= \boxed{1.41667} \\ x_3 &= \boxed{1.41421} \end{aligned}$$

Verification: $\sqrt{2} = 1.41421356...$, so x_3 is accurate to 5 decimal places after just 3 iterations!

Observation: The method converges very quickly - this is called ”quadratic convergence.”

Example 17: Newton-Raphson with Four Decimal Places

Problem: Use Newton-Raphson method to approximate a root of $e^x - 3x = 0$ up to four decimal places, starting with $x_0 = 0$.

(From 2024 past paper, Q3 part a)

Solution:

Step 1: Set up the function and derivative

$$\begin{aligned} f(x) &= e^x - 3x \\ f'(x) &= e^x - 3 \end{aligned}$$

Step 2: Write iteration formula

$$\begin{aligned} x_{n+1} &= x_n - \frac{f(x_n)}{f'(x_n)} \\ &= x_n - \frac{e^{x_n} - 3x_n}{e^{x_n} - 3} \end{aligned}$$

Step 3: Perform iterations starting with $x_0 = 0$

Iteration 0: $x_0 = 0$

Iteration 1:

$$\begin{aligned} f(0) &= e^0 - 3(0) = 1 - 0 = 1 \\ f'(0) &= e^0 - 3 = 1 - 3 = -2 \\ x_1 &= 0 - \frac{1}{-2} = 0 - (-0.5) = 0.5 \end{aligned}$$

Iteration 2:

$$\begin{aligned} f(0.5) &= e^{0.5} - 3(0.5) = 1.6487 - 1.5 = 0.1487 \\ f'(0.5) &= e^{0.5} - 3 = 1.6487 - 3 = -1.3513 \\ x_2 &= 0.5 - \frac{0.1487}{-1.3513} = 0.5 + 0.1100 = 0.6100 \end{aligned}$$

Iteration 3:

$$\begin{aligned} f(0.6100) &= e^{0.61} - 3(0.61) = 1.8404 - 1.83 = 0.0104 \\ f'(0.6100) &= e^{0.61} - 3 = 1.8404 - 3 = -1.1596 \\ x_3 &= 0.61 - \frac{0.0104}{-1.1596} = 0.61 + 0.0090 = 0.6190 \end{aligned}$$

Iteration 4:

$$\begin{aligned} f(0.6190) &= e^{0.619} - 3(0.619) = 1.8572 - 1.857 = 0.0002 \\ f'(0.6190) &= e^{0.619} - 3 = 1.8572 - 3 = -1.1428 \\ x_4 &= 0.619 - \frac{0.0002}{-1.1428} = 0.619 + 0.0002 = 0.6192 \end{aligned}$$

Iteration 5:

$$x_5 \approx 0.6192$$

Since x_4 and x_5 agree to 4 decimal places, we stop.

Answer: The root is approximately $\boxed{x \approx 0.6192}$ (to 4 decimal places)

Verification: $f(0.6192) = e^{0.6192} - 3(0.6192) \approx 0.000003 \approx 0$ ✓

Example 18: Finding Multiple Roots

Problem: The equation $x^3 - 2x - 5 = 0$ has one real root. Use Newton-Raphson method with $x_0 = 2$ to approximate it to 4 decimal places.

Solution:
Step 1: Set up the functions

$$f(x) = x^3 - 2x - 5$$
$$f'(x) = 3x^2 - 2$$

Step 2: Iteration formula

$$x_{n+1} = x_n - \frac{x_n^3 - 2x_n - 5}{3x_n^2 - 2}$$

Step 3: Perform iterations
$x_0 = 2$:
Iteration 1:

$$f(2) = 8 - 4 - 5 = -1$$
$$f'(2) = 12 - 2 = 10$$
$$x_1 = 2 - \frac{-1}{10} = 2.1$$

Iteration 2:

$$f(2.1) = (2.1)^3 - 2(2.1) - 5 = 9.261 - 4.2 - 5 = 0.061$$
$$f'(2.1) = 3(2.1)^2 - 2 = 13.23 - 2 = 11.23$$
$$x_2 = 2.1 - \frac{0.061}{11.23} = 2.1 - 0.0054 = 2.0946$$

Iteration 3:

$$f(2.0946) \approx 0.0003$$
$$f'(2.0946) \approx 11.1575$$
$$x_3 = 2.0946 - \frac{0.0003}{11.1575} \approx 2.0946$$

Convergence achieved!
Answer: $\boxed{x \approx 2.0946}$
Check: $f(2.0946) = (2.0946)^3 - 2(2.0946) - 5 \approx 0.0001 \approx 0$ ✓

Definition 3.8: When Newton-Raphson Method Fails

The Newton-Raphson method may fail or converge slowly when:

1. Poor Initial Guess:

- x_0 is far from the actual root
- May converge to a different root
- May oscillate or diverge

2. $f'(x_n) = 0$ or very small:

- Division by zero or near-zero
- Tangent line nearly horizontal
- Next iterate may jump very far

3. Multiple Roots:

- At multiple roots, $f'(x) = 0$
- Convergence is slower (linear instead of quadratic)

4. Cycles:

- Iterations may cycle between values
- Never converge to a root

How to Improve:

- Choose better initial guess (graph the function first)
- Use a different method if Newton-Raphson fails
- Check for convergence (iterations should get closer)

Example 19: Convergence Analysis

Problem: For $f(x) = x^2 - 4$, starting with $x_0 = 1$, show the first three Newton-Raphson iterations and discuss convergence.

Solution:
Step 1: Set up

$$f(x) = x^2 - 4$$
$$f'(x) = 2x$$
$$x_{n+1} = x_n - \frac{x_n^2 - 4}{2x_n} = \frac{x_n^2 + 4}{2x_n}$$

Step 2: Iterations
$x_0 = 1$**:**

$$x_1 = \frac{1+4}{2} = 2.5$$

$x_1 = 2.5$**:**

$$x_2 = \frac{6.25+4}{5} = \frac{10.25}{5} = 2.05$$

$x_2 = 2.05$**:**

$$x_3 = \frac{(2.05)^2+4}{2(2.05)} = \frac{4.2025+4}{4.1} = \frac{8.2025}{4.1} \approx 2.0006$$

Step 3: Analysis
The sequence: $1 \to 2.5 \to 2.05 \to 2.0006 \to \ldots$
Converging to $x = 2$ (since $f(2) = 4 - 4 = 0$).
Convergence table:

Iteration	x_n	Error $\lvert x_n - 2\rvert$
0	1.0000	1.0000
1	2.5000	0.5000
2	2.0500	0.0500
3	2.0006	0.0006

Answer: The method converges **rapidly** (quadratically) to the root $x = 2$.
Note: Each iteration roughly squares the error, which is characteristic of quadratic convergence.

Section 3.4 Summary: Newton-Raphson Method

The Formula:

$$\boxed{x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}}$$

Algorithm Steps:

1. Choose initial guess x_0 (close to actual root)
2. Compute $f(x_n)$ and $f'(x_n)$
3. Calculate $x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}$
4. Repeat until desired accuracy achieved
5. Stop when $|x_{n+1} - x_n| < \epsilon$ (tolerance)

Advantages:

- Very fast convergence (quadratic)
- Works for many types of equations
- Simple to implement

Disadvantages:

- Requires derivative
- May not converge with poor initial guess
- Fails when $f'(x_n) = 0$
- May converge to wrong root

Important Tips:

- Always check convergence (iterations getting closer)
- Graph the function to find good starting point
- Stop when successive iterates agree to desired decimal places
- Be careful with rounding during calculations
- Verify your answer by substituting into original equation

Exam Pattern:

Past papers typically ask for:

- 3-5 iterations
- Accuracy to 4 decimal places
- Show all working clearly
- State the iteration formula first

Practice Exercises - Chapter 3

Problem 1: Basic Parametric Derivatives

Find $\dfrac{dy}{dx}$ for each parametric curve:

(a) $x = t^2 - 1, \quad y = t^3 + t$

(b) $x = 3\cos t, \quad y = 3\sin t$

(c) $x = e^t, \quad y = e^{-t}$

(d) $x = t - \sin t, \quad y = 1 - \cos t$

(e) $x = \ln t, \quad y = t^2 + 1$

Problem 2: Parametric Derivative at a Point

(a) Find $\dfrac{dy}{dx}$ at $t = \dfrac{\pi}{4}$ for the curve:

$$x = \sin t, \quad y = \cos t$$

(b) Find the slope of the tangent to:

$$x = t^2 + 1, \quad y = t^3 - 3t$$

at $t = 2$.

(c) Find $\dfrac{dy}{dx}$ at $t = 1$ for:

$$x = 2t - 1, \quad y = t^2 + 3t - 2$$

(d) For the curve $x = e^t \cos t$, $y = e^t \sin t$, find the slope at $t = 0$.

Problem 3: Second Derivative of Parametric Equations

(a) Find $\dfrac{d^2y}{dx^2}$ for:

$$x = t^2, \quad y = t^3$$

(b) Find $\dfrac{d^2z}{dx^2}$ as a function of t if:

$$x = t + \frac{1}{t}, \quad z = t - \frac{1}{t}$$

(From 2024 past paper, Q2 part b)

(c) Find $\dfrac{d^2y}{dx^2}$ for the parametric curve:

$$x = \cos t, \quad y = \sin t$$

and verify that it gives the correct concavity for the unit circle.

Problem 4: Tangent Lines to Parametric Curves

(a) Find the equation of the tangent line to:

$$x = \cos^3 t, \quad y = \sin^3 t$$

at $t = \frac{\pi}{4}$.

(b) Find the equation of the tangent line to:

$$x = t^2, \quad y = t^3 - t$$

at the point where $t = 2$.

(c) For the curve $x = 1 + \sqrt{t}$, $y = e^{t^2}$, find the tangent line at $t = 0$.

(d) Find all values of t where the tangent line to:

$$x = t - \sin t, \quad y = 1 - \cos t$$

is horizontal.

Problem 5: Arc Length of Parametric Curves

(a) Find the arc length of the curve:

$$x = \cos^3 t, \quad y = \sin^3 t, \quad 0 \le t \le \pi$$

(b) Find the length of the curve:

$$x = t\cos t, \quad y = t\sin t, \quad 0 \le t \le 1$$

(c) For the parametric curve:

$$x = 3t^2, \quad y = 2t^3, \quad 0 \le t \le 1$$

Set up (but do not necessarily evaluate) the arc length integral.

(d) Show that the circumference of a circle of radius r given by:

$$x = r\cos t, \quad y = r\sin t, \quad 0 \le t \le 2\pi$$

is $2\pi r$.

Problem 6: Surface Area of Parametric Curves

(a) Find the area of the surface generated by revolving the curve:

$$y = 2\sqrt{x}, \quad 1 \le x \le 2$$

about the x-axis.

(From 2022 past paper, Q3 part a)

(b) For the astroid $x = \cos^3 t$, $y = \sin^3 t$, find $\frac{dy}{dx}$ and simplify as much as possible.

(c) Find the length of the astroid:

$$x = \cos^3 t, \quad y = \sin^3 t, \quad 0 \le t \le \pi$$

Set up the integral and evaluate.

Problem 7: Expanding and Shrinking Shapes

(a) The radius of a circle is increasing at 3 cm/s. How fast is the area increasing when the radius is 8 cm?

(b) A square's side length is decreasing at 2 m/s. How fast is the area decreasing when the side is 10 m?

(c) The volume of a cube is increasing at 300 cm^3/s. How fast is the side length increasing when the side is 10 cm?

(d) A circular oil slick is expanding. Its radius is increasing at 2 m/min. How fast is the area of the oil slick increasing when the radius is 50 m?

Problem 8: Spheres and Balloons

(a) A spherical balloon is being inflated at 100 cm^3/s. How fast is the radius increasing when the radius is 5 cm?

(b) A snowball is melting. Its volume is decreasing at 1 cm^3/min. How fast is the radius decreasing when the radius is 2 cm?

(c) A spherical ball bearing is manufactured with radius 0.5 cm. If the radius can vary by at most 0.02 cm, by how much can the volume vary? (Use differentials)

(d) The surface area of a sphere is $S = 4\pi r^2$. If the radius is increasing at 2 cm/s, how fast is the surface area increasing when $r = 3$ cm?

Problem 9: Ladder and Wall Problems

(a) A 13-meter ladder leans against a wall. The bottom slides away at 2 m/s. How fast is the top sliding down when the bottom is 5 m from the wall?

(b) A ladder 10 meters long leans against a vertical wall. The bottom is sliding away at 1 m/s. Find the rate at which:

- The top is sliding down when the bottom is 6 m from the wall
- The angle between the ladder and the wall is changing at the same instant

(c) A person 1.8 m tall walks away from a 6 m lamppost at 1.2 m/s. How fast is the tip of their shadow moving?

Problem 10: Conical and Cylindrical Tanks

(a) Water drains from a conical tank (height 10 m, radius 5 m at top) at 2 m^3/min. How fast is the water level falling when the water is 6 m deep?

(b) Water fills a cylindrical tank (radius 3 m) at 2 m^3/min. How fast is the water level rising?

(c) A conical paper cup (height 10 cm, radius 3 cm at top) is being filled at 2 cm^3/s. How fast is the water level rising when the water is 5 cm deep?

(d) Sand falls onto a conical pile at 10 cm^3/s. The height always equals the diameter. How fast is the height increasing when $h = 5$ cm?

Problem 11: Moving Objects and Distances

(a) Two cars approach an intersection. Car A moves north at 60 km/h and car B moves west at 80 km/h. When car A is 0.5 km south and car B is 0.6 km east of the intersection, how fast is the distance between them changing?

(b) A boat moves away from a dock at 3 m/s. A rope attached to the bow passes over a pulley 4 m above the dock. How fast is the rope being pulled when the boat is 3 m from below the pulley?

(c) Two particles move on the x-axis. Particle A has position $x_A = t^2$ and particle B has position $x_B = 2t + 5$. At what rate is the distance between them changing at $t = 3$?

(d) A kite is 100 m above the ground. The wind carries it horizontally at 8 m/s. How fast is the string being let out when the string is 125 m long?

Problem 12: Angles and Trigonometric Rates

(a) A searchlight rotates at 2 revolutions/minute. A wall is 100 m from the searchlight. How fast is the light beam moving along the wall when the beam makes a 45 angle with the wall?

(b) The angle of elevation of the sun decreases at a rate of 0.25/min. How fast does the shadow of a 10-meter pole lengthen when the elevation is 60?

(c) A camera tracks a rocket launched vertically. The camera is 1000 m from the launch pad. The rocket rises at 300 m/s. How fast must the camera angle increase when the rocket is 1000 m high?

Problem 13: Computing Higher Derivatives

Find the second derivative $f''(x)$ for:

(a) $f(x) = x^5 - 3x^4 + 2x - 7$

(b) $f(x) = x^2 \sin x$

(c) $f(x) = \dfrac{x}{x^2 + 1}$

(d) $f(x) = \ln(x^2 + 1)$

(e) $f(x) = e^x \cos x$

(f) $f(x) = \sqrt{x^2 + 4}$

Problem 14: Implicit Second Derivatives

Find $\dfrac{d^2y}{dx^2}$ using implicit differentiation:

(a) $x^2 + y^2 = 25$

(b) $x^2 + y^4 = 4$

(c) $x^2 + y^4 = 16$

(From 2025 past paper, Q3)

(d) $xy + y^2 = 3$

(e) $\sin(x + y) = x$

(Show all steps including finding $\dfrac{dy}{dx}$ first)

Problem 15: Velocity Acceleration and Jerk

(a) The position of a particle is $s(t) = t^4 - 8t^3 + 24t^2 - 32t + 10$ meters.

- Find velocity $v(t)$, acceleration $a(t)$, and jerk $j(t)$
- When is the particle at rest?
- When is acceleration zero?
- When is jerk zero?

(b) What is jerk? Why is the jerk during free fall zero? Explain fully.
(From 2024 past paper, Q1 part v)

(c) A particle moves with acceleration $a(t) = 3t - 2$ m/s². If $v(0) = 4$ m/s and $s(0) = 1$ m, find the position function $s(t)$.

(d) During free fall, a particle has position $s(t) = -4.9t^2 + 20t + 50$ m. Find velocity and acceleration, and confirm that jerk is zero.

Problem 16: Concavity and Inflection Points

For each function, find:

- Intervals where the function is concave up
- Intervals where the function is concave down
- All inflection points

(a) $f(x) = x^3 - 3x^2 + 4$
(b) $f(x) = 2x^3 - 3x^2 - 12x$ (From 2024 past paper, Q2)
(c) $f(x) = x^4 - 4x^3$
(d) $h(x) = -x^3 + 2x^2$ (From 2023 past paper, Q3 part b - increasing/decreasing)
(e) $f(x) = x^2 e^{-x}$
(f) $f(x) = \ln(x^2 + 1)$

Problem 17: Increasing and Decreasing Functions

Find the intervals where the following functions are increasing and decreasing:
(a) $f(x) = x^3(4 - x)$ (From 2022 past paper, Q5 part a)
(b) $h(x) = -x^3 + 2x^2$ (From 2023 past paper, Q3 part b)
(c) $f(x) = 2x^3 - 3x^2 - 12x + 1$ (From 2024 past paper Q5 - related)
(d) $g(x) = x^4 - 4x^3 + 4x^2$
(e) $f(x) = xe^{-x}$
(f) $f(x) = \dfrac{x^2}{x^2 + 1}$

Problem 18: Second Derivative Test

Use the second derivative test to find and classify all local extrema:
(a) $f(x) = x^3 - 6x^2 + 9x + 1$
(b) $f(x) = 2x^3 - 3x^2 - 12x + 1$

Find:

- All critical points
- Local maxima and minima
- Inflection points
- Intervals of concavity

(c) $f(x) = x^4 - 2x^2 + 3$
(d) $f(x) = xe^{-x}$
(e) $f(x) = x + \dfrac{4}{x}$ on $(0, \infty)$

Problem 19: Newton-Raphson – Standard Applications

Use Newton-Raphson method to find the indicated root to 4 decimal places:
(a) $x^2 - 3 = 0$ (i.e., approximate $\sqrt{3}$), with $x_0 = 1.5$.
Show the iteration formula and at least 4 iterations.
(b) $x^3 - 2 = 0$ (i.e., approximate $\sqrt[3]{2}$), with $x_0 = 1$.
(c) $e^x - 3x = 0$ with $x_0 = 0$.
(From 2024 past paper, Q3 part a)
(d) $x^3 - 2x - 5 = 0$ with $x_0 = 2$.

Problem 20: Newton-Raphson – Trigonometric and Exponential Equations

Use Newton-Raphson method to approximate the root to 4 decimal places:
(a) $\cos x = x$ with $x_0 = 1$ (find the fixed point).
(b) $e^x + x = 3$ with $x_0 = 0$.
(c) $\sin x = x - 1$ with $x_0 = 2$.
(d) $\ln x + x = 2$ with $x_0 = 1$.
Show clearly:

- The function $f(x)$ and $f'(x)$
- The iteration formula
- At least 4 iterations in a table
- Final approximate answer

Problem 21: Newton-Raphson – Error Analysis and Convergence

(a) For $f(x) = x^2 - 2$ with $x_0 = 1$, perform 5 iterations and compare with exact answer $\sqrt{2} = 1.41421356...$

Calculate the absolute error at each step and observe the rate of convergence.

(b) Starting with $x_0 = 1$, apply Newton-Raphson to $f(x) = x^3 - x - 1$. Show 4 iterations.

(c) For $f(x) = x^2$, explain why Newton-Raphson converges slowly (or fails) to the root $x = 0$.

(d) For $f(x) = x^3 - 3x^2 + 3x - 1$, the root is $x = 1$ (triple root). Starting with $x_0 = 0.5$, show 5 iterations and explain why convergence is slow.

Problem 22: Newton-Raphson – Setting Up and Solving

(a) Show that the Newton-Raphson formula for finding $\sqrt[n]{a}$ (i.e., solving $x^n - a = 0$) is:

$$x_{n+1} = \frac{(n-1)x_n^n + a}{n \cdot x_n^{n-1}}$$

(b) Use the formula from part (a) with $n = 3$ to find $\sqrt[3]{10}$ to 4 decimal places. Start with $x_0 = 2$.

(c) The equation $\ln x = \dfrac{1}{x}$ has a root near $x = 1.76$. Use Newton-Raphson with $x_0 = 1.76$ to refine to 4 decimal places.

(d) Show that if $f(x) = x^2 - c$ for some constant $c > 0$, then Newton-Raphson gives:

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{c}{x_n}\right)$$

This is the **Babylonian method** for computing square roots!

Problem 23: Mixed Differentiation Techniques

(a) A particle moves with position $s(t) = \sin^2 t$. Find velocity and acceleration.

(b) For the parametric curve $x = t^2$, $y = t^3$:

- Find $\dfrac{dy}{dx}$ and $\dfrac{d^2y}{dx^2}$
- Find the equation of the tangent line at $t = 1$
- Find the inflection point(s) of the curve

(c) Water fills a hemispherical bowl of radius R at rate Q m³/s. If the volume of water when height is h is $V = \pi h^2\left(R - \dfrac{h}{3}\right)$, find $\dfrac{dh}{dt}$ in terms of h, R, and Q.

(d) The radius of curvature of a curve is defined as:

$$\kappa = \left|\frac{d^2y/dx^2}{[1 + (dy/dx)^2]^{3/2}}\right|$$

Find the curvature of $y = x^2$ at the origin.
(From 2022 past paper, Q1 part vi – curvature concept)

Problem 24: Proving Identities Using Derivatives

(a) If $f(x) = \ln(1 + \sqrt{1-x})$, prove that:

$$4x(1-x)f''(x) + 2(2-3x)f'(x) + 1 = 0$$

(From 2024 past paper, Q6 part a)

(b) If $y = A\sin(\omega t) + B\cos(\omega t)$ where A, B, and ω are constants, show that:

$$\frac{d^2y}{dt^2} + \omega^2 y = 0$$

This is the equation of simple harmonic motion!
(c) If $y = e^{ax}\cos(bx)$, find y' and y'', then show:

$$y'' - 2ay' + (a^2 + b^2)y = 0$$

(d) For $y = x^n$ where n is a positive integer, show that:

$$y^{(n)} = n! \quad \text{(constant)}$$

and

$$y^{(n+1)} = y^{(n+2)} = \cdots = 0$$

Problem 25: Comprehensive Challenge Problems

(a) A ladder 10 m long rests against a vertical wall. Let θ be the angle the ladder makes with the floor. If the bottom slides at 1 m/s, how fast is θ decreasing when the bottom is 5 m from the wall?

(b) Use the parametric form of the circle ($x = r\cos t$, $y = r\sin t$) and parametric differentiation to find the slope of the tangent to the circle at the point $(r\cos t_0, r\sin t_0)$. Confirm this matches the result obtained by implicit differentiation of $x^2 + y^2 = r^2$.

(c) A particle moves along the curve $y = x^3 - 3x$. At what points does the velocity vector make a 45 angle with the x-axis? (Hint: slope $= \tan 45 = 1$)

(d) Apply Newton-Raphson method to find the root of $f(x) = x^3 - x - 1$ starting with:

- $x_0 = 1.5$: Show 4 iterations
- $x_0 = -1$: Show what happens (does it converge?)

Compare the two cases and discuss what this tells us about the importance of initial guess.

(e) For the curve given parametrically by:

$$x = 2\cos t, \quad y = \sin(2t)$$

- Find $\dfrac{dy}{dx}$
- Find all t values where the tangent is horizontal
- Find all t values where the tangent is vertical
- Find the Cartesian equation of this curve by eliminating t

Chapter 3 Practice Problem Summary

Topic Coverage:

Topic	Problems	Exam Relevance
Parametric Differentiation	1–6	Medium
Related Rates	7–12	Medium
Higher Derivatives	13–15	High
Concavity & Inflection	16–18	Very High
Newton-Raphson	19–22	High
Mixed & Proofs	23–25	Advanced

Past Paper Connections:

Past Paper	Related Problems
2021	Problem 2(a), Problem 4(a)
2022	Problem 5(a), Problem 6(a), Problem 17(a)
2023	Problem 14(c), Problem 16(d), Problem 17(b)
2024	Problem 3(b), Problem 15(b), Problem 16(b), Problem 19(c), Problem 24(a)
2025	Problem 14(c), Problem 17(c)

Study Strategy for Chapter 3:

1. **Parametric (Problems 1–6):** Practice converting to Cartesian when possible to verify answers
2. **Related Rates (Problems 7–12):** Always draw a diagram! Label all variables before writing equations
3. **Higher Derivatives (Problems 13–18):** Remember the sign chart method for concavity
4. **Newton-Raphson (Problems 19–22):** Practice showing work in table format (as expected in exams)
5. **Mixed Problems (23–25):** These require combining multiple techniques

Most Frequently Tested in Exams:

- **Concavity and inflection points** (appears every year!)
- **Increasing/decreasing intervals** (appears every year!)
- **Second derivative test** (very frequent)
- **Newton-Raphson** (appears most years)
- **Higher derivatives and proofs** (frequently)
- **Related rates** (appears occasionally)
- **Parametric differentiation** (appears occasionally)

Self-Assessment Checklist:

Before the exam, make sure you can:

☐ Find $\dfrac{dy}{dx}$ and $\dfrac{d^2y}{dx^2}$ for parametric equations

☐ Set up and solve related rates problems from a diagram

☐ Find and classify concavity intervals and inflection points

☐ Apply second derivative test to classify extrema

☐ Perform Newton-Raphson iterations to desired accuracy

☐ Prove differential identities using higher derivatives

☐ Interpret velocity, acceleration, and jerk physically

Final Exam Preparation Tip:
Review the following worked examples from Chapter 3 before your exam:

- **Example 3:** Second derivative of parametric equations (2024 paper)
- **Example 9:** Water tank related rates
- **Example 14:** Finding inflection points (2024 paper)
- **Example 17:** Newton-Raphson with $e^x - 3x = 0$ (2024 paper)
- **Example 19:** Convergence analysis

Chapter 4

Applications of Differentiation

Extreme Value Theorem

Finding maximum and minimum values of functions is one of the most important applications of calculus. These concepts appear in optimization problems across physics, engineering, economics, and everyday decision-making.

Definition 4.1: Absolute (Global) Extrema

Let f be a function defined on a domain D.

Absolute Maximum: $f(c)$ is the **absolute maximum** of f on D if:

$$\boxed{f(c) \geq f(x) \quad \text{for all } x \in D}$$

Absolute Minimum: $f(c)$ is the **absolute minimum** of f on D if:

$$\boxed{f(c) \leq f(x) \quad \text{for all } x \in D}$$

Key Terminology:

- Absolute maximum and minimum together are called **absolute extrema** or **global extrema**
- The extremum is a **value** (the y-coordinate), not the point itself
- A function may have only one absolute maximum and one absolute minimum
- Some functions may not attain absolute extrema if the domain is not closed or bounded

Important: The absolute maximum *value* is the largest y-value, not the x-value where it occurs!

Definition 4.2: Local (Relative) Extrema

Local Maximum: $f(c)$ is a **local maximum** if there exists an open interval (a, b) containing c such that:

$$f(c) \geq f(x) \quad \text{for all } x \in (a, b)$$

Local Minimum: $f(c)$ is a **local minimum** if there exists an open interval (a, b) containing c such that:

$$f(c) \leq f(x) \quad \text{for all } x \in (a, b)$$

Key Differences:

Local Extrema	Absolute Extrema
Highest/lowest in a neighbourhood	Highest/lowest on entire domain
Can have many local max/min	At most one absolute max/min
Every absolute extremum is local	Not every local extremum is absolute

Theorem 4.1: The Extreme Value Theorem (EVT)

If f is **continuous** on a **closed interval** $[a, b]$, then f attains both an absolute maximum and an absolute minimum on $[a, b]$.

That is, there exist numbers c and d in $[a, b]$ such that:

$$\boxed{f(c) \leq f(x) \leq f(d) \quad \text{for all } x \in [a, b]}$$

where $f(c)$ is the absolute minimum and $f(d)$ is the absolute maximum.

Two Essential Conditions:

1. f must be **continuous** on $[a, b]$
2. The interval must be **closed** (includes both endpoints a and b)

If Either Condition Fails, EVT May Not Hold:

- $f(x) = \dfrac{1}{x}$ on $(0, 1)$: open interval $\to$ no minimum exists
- $f(x) = \tan x$ on $[0, \frac{\pi}{2}]$: discontinuous at $\frac{\pi}{2}$ $\to$ no maximum exists
- $f(x) = x$ on $(-\infty, \infty)$: unbounded domain $\to$ no absolute extrema

Definition 4.3: Critical Numbers

A number c in the domain of f is called a **critical number** (or critical point) of f if either:

$$\boxed{f'(c) = 0} \quad \textbf{OR} \quad \boxed{f'(c) \text{ does not exist}}$$

Fermat's Theorem:

If f has a local maximum or minimum at c, and $f'(c)$ exists, then $f'(c) = 0$.
Contrapositive: If $f'(c) \neq 0$, then f does not have a local extremum at c.

Converse is FALSE:

Not every critical number gives a local extremum! For example, $f(x) = x^3$ has $f'(0) = 0$, but $x = 0$ is neither a max nor a min.

Types of Critical Numbers:

- $f'(c) = 0$: horizontal tangent (stationary point)
- $f'(c)$ undefined: corner, cusp, or vertical tangent

Definition 4.4: The Closed Interval Method

To find the **absolute maximum and minimum** of a continuous function f on a closed interval $[a, b]$:

Step 1: Find all critical numbers of f in the *open* interval (a, b)

- Solve $f'(x) = 0$
- Find where $f'(x)$ does not exist

Step 2: Evaluate f at each critical number found in Step 1
Step 3: Evaluate f at the endpoints: $f(a)$ and $f(b)$
Step 4: Compare all values from Steps 2 and 3:

- The **largest** value is the absolute maximum
- The **smallest** value is the absolute minimum

⚠ **Critical Warning:**
NEVER forget the endpoints! Absolute extrema can occur at endpoints even when there are critical numbers inside the interval.

Example 1: Finding Critical Numbers

Problem: Find all critical numbers of:
(a) $f(x) = 2x^3 - 3x^2 - 12x + 5$
(b) $g(x) = x^{2/3}$
(c) $h(x) = |x - 2|$

Solution:
(a) $f(x) = 2x^3 - 3x^2 - 12x + 5$
Find $f'(x)$:

$$\begin{aligned} f'(x) &= 6x^2 - 6x - 12 \\ &= 6(x^2 - x - 2) \\ &= 6(x - 2)(x + 1) \end{aligned}$$

Set $f'(x) = 0$:

$$6(x - 2)(x + 1) = 0 \implies x = 2 \text{ or } x = -1$$

$f'(x)$ exists for all x.
Critical numbers: $\boxed{x = 2, \quad x = -1}$

(b) $g(x) = x^{2/3}$
Find $g'(x)$:

$$g'(x) = \frac{2}{3}x^{-1/3} = \frac{2}{3x^{1/3}}$$

$g'(x) = 0$: Never (numerator is always 2)
$g'(x)$ undefined: When $3x^{1/3} = 0$, i.e., when $x = 0$
Critical number: $\boxed{x = 0}$ (where derivative doesn't exist)

(c) $h(x) = |x - 2|$

Rewrite: $h(x) = \begin{cases} -(x - 2) & x < 2 \\ x - 2 & x \geq 2 \end{cases}$

Therefore: $h'(x) = \begin{cases} -1 & x < 2 \\ 1 & x > 2 \\ \text{undefined} & x = 2 \end{cases}$

$h'(x) = 0$: Never
$h'(x)$ undefined: At $x = 2$
Critical number: $\boxed{x = 2}$ (corner point)

Example 2: Closed Interval Method

Problem: Find the absolute maximum and minimum values of $f(x) = x^2 - 1$ on $[-1, 2]$. (From 2021 past paper, Q4 part b)

Solution:
Step 1: Find critical numbers in $(-1, 2)$

$$f'(x) = 2x$$
$$f'(x) = 0 \implies x = 0$$

Check: $0 \in (-1, 2)$ ✓
Step 2: Evaluate at critical number

$$f(0) = 0^2 - 1 = -1$$

Step 3: Evaluate at endpoints

$$f(-1) = (-1)^2 - 1 = 1 - 1 = 0$$
$$f(2) = 2^2 - 1 = 4 - 1 = 3$$

Step 4: Compare all values

x	$f(x)$	**Type**
-1	0	Endpoint
0	-1	Critical Number
2	3	Endpoint

Answer:

- **Absolute Maximum:** $\boxed{f(2) = 3 \text{ at } x = 2}$
- **Absolute Minimum:** $\boxed{f(0) = -1 \text{ at } x = 0}$

Geometric Check: This is a parabola $y = x^2 - 1$ opening upward with vertex at $(0, -1)$. On $[-1, 2]$, the minimum occurs at the vertex and the maximum at the rightmost endpoint. ✓

Example 3: Absolute Extrema – Polynomial

Problem: Find the absolute maximum and minimum values of $f(x) = 2x^3 - 3x^2 - 12x + 1$ on $[-2, 3]$.

(From 2025 past paper, Q4)

Solution:

Step 1: Find critical numbers

$$\begin{aligned} f'(x) &= 6x^2 - 6x - 12 \\ &= 6(x^2 - x - 2) \\ &= 6(x-2)(x+1) \end{aligned}$$

Setting $f'(x) = 0$:

$$x = 2 \quad \text{or} \quad x = -1$$

Both $x = 2$ and $x = -1$ lie in $(-2, 3)$ ✓

Step 2: Evaluate at critical numbers

$$\begin{aligned} f(2) &= 2(8) - 3(4) - 12(2) + 1 \\ &= 16 - 12 - 24 + 1 \\ &= -19 \end{aligned}$$

$$\begin{aligned} f(-1) &= 2(-1) - 3(1) - 12(-1) + 1 \\ &= -2 - 3 + 12 + 1 \\ &= 8 \end{aligned}$$

Step 3: Evaluate at endpoints

$$\begin{aligned} f(-2) &= 2(-8) - 3(4) - 12(-2) + 1 \\ &= -16 - 12 + 24 + 1 \\ &= -3 \end{aligned}$$

$$\begin{aligned} f(3) &= 2(27) - 3(9) - 12(3) + 1 \\ &= 54 - 27 - 36 + 1 \\ &= -8 \end{aligned}$$

Step 4: Compare all values

x	$f(x)$	**Type**
-2	-3	Endpoint
-1	8	Critical Number
2	-19	Critical Number
3	-8	Endpoint

Answer:

- **Absolute Maximum:** $\boxed{f(-1) = 8 \text{ at } x = -1}$
- **Absolute Minimum:** $\boxed{f(2) = -19 \text{ at } x = 2}$

Verification: Maximum value: 8 is largest among $\{-3, 8, -19, -8\}$ ✓
Minimum value: -19 is smallest ✓

Section 4.1 Summary

Extreme Value Theorem:

$$f \text{ continuous on } [a,b] \implies f \text{ attains absolute max and min on } [a,b]$$

Closed Interval Method – Finding Absolute Extrema:

1. Find all critical numbers in (a,b): solve $f'(x) = 0$ and find where f' is undefined
2. Evaluate f at each critical number
3. Evaluate f at both endpoints a and b
4. Largest value = Absolute Maximum
5. Smallest value = Absolute Minimum

Most Common Mistakes:

- **Forgetting to check endpoints** – absolute extrema often occur there!
- Confusing local extrema with absolute extrema
- Not verifying that critical numbers lie inside (a,b)
- Stating the x-coordinate instead of the y-value as the extremum

Quick Memory Aid: EVT = Closed + Continuous $\implies$ Extrema Exist

Rolle's Theorem

Rolle's Theorem is a special case of the Mean Value Theorem and provides important theoretical foundations for understanding function behavior. Though it appears simple, it has profound implications throughout calculus.

Theorem 4.2: Rolle's Theorem

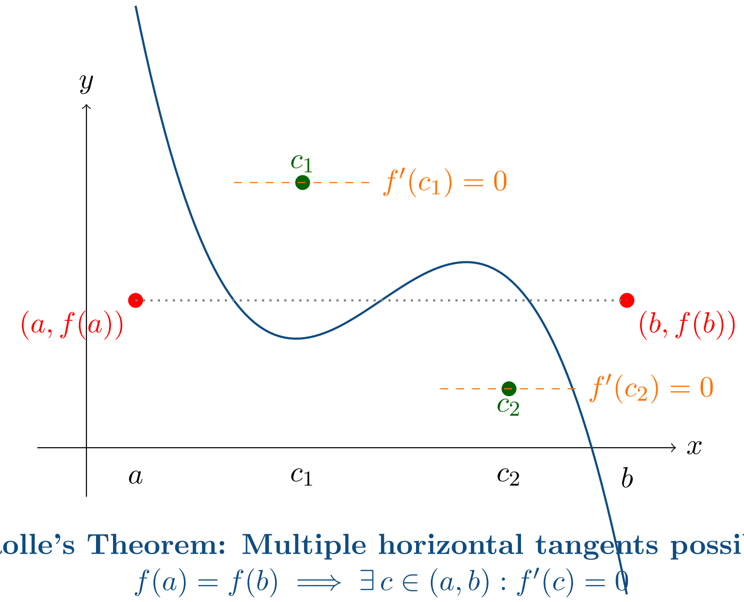
If a function f satisfies **all three** conditions:

1. f is **continuous** on the closed interval $[a, b]$
2. f is **differentiable** on the open interval (a, b)
3. $f(a) = f(b)$ (equal function values at endpoints)

Then there exists at least one number c in the open interval (a, b) such that:

$$\boxed{f'(c) = 0}$$

Geometric Interpretation:
If a smooth curve starts and ends at the same height, there must be at least one point in between where the tangent line is horizontal (slope = 0).



Rolle's Theorem: Multiple horizontal tangents possible
$f(a) = f(b) \implies \exists c \in (a, b) : f'(c) = 0$

Definition 4.5: Conditions for Rolle's Theorem

All three conditions are **essential**:

Condition 1 – Continuity on $[a, b]$:
Ensures the function has no gaps or jumps. Without this, the function might not attain all values between $f(a)$ and $f(b)$.

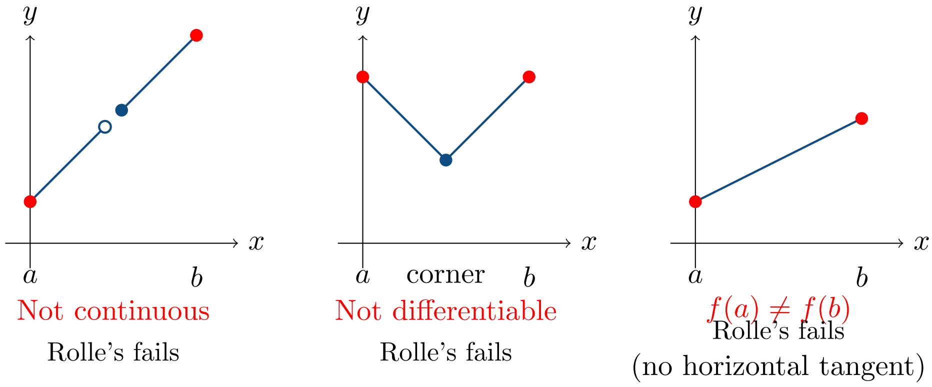
Condition 2 – Differentiability on (a, b):
Ensures the function has a well-defined tangent line everywhere inside the interval. Corners or cusps would violate this.

Condition 3 – Equal Endpoint Values:

$f(a) = f(b)$ ensures the function returns to the same height, guaranteeing at least one turning point (horizontal tangent) in between.

⚠ **If Any Condition Fails:**

Rolle's Theorem may not apply, and there may be no horizontal tangent in (a, b).



Example 4: Verifying Rolle's Theorem

Problem: Verify that Rolle's Theorem applies to $f(x) = x^2 - 4x + 3$ on $[1, 3]$, and find all values of c guaranteed by the theorem.

Solution:

Step 1: Verify all three conditions

Condition 1 (Continuous): $f(x) = x^2 - 4x + 3$ is a polynomial, so continuous everywhere, including on $[1, 3]$. ✓

Condition 2 (Differentiable): Polynomials are differentiable everywhere, including on $(1, 3)$. ✓

Condition 3 (Equal endpoints):

$$f(1) = 1 - 4 + 3 = 0$$
$$f(3) = 9 - 12 + 3 = 0$$

So $f(1) = f(3) = 0$ ✓

All three conditions satisfied! Rolle's Theorem guarantees at least one $c \in (1, 3)$ with $f'(c) = 0$.

Step 2: Find c

$$\begin{aligned} f'(x) &= 2x - 4 \\ f'(c) &= 0 \\ 2c - 4 &= 0 \\ c &= 2 \end{aligned}$$

Step 3: Verify $c \in (1, 3)$

$c = 2$ and $1 < 2 < 3$ ✓

Answer: $\boxed{c = 2}$

Geometric Interpretation: The parabola $y = x^2 - 4x + 3$ has its vertex at $x = 2$. At this point, the tangent line is horizontal, which is exactly what Rolle's Theorem predicts!

Example 5: Rolle's Theorem – Trigonometric Function

Problem: Verify Rolle's Theorem for $f(x) = \sin x$ on $[0, \pi]$ and find all values of c.

Solution:
Step 1: Verify conditions
Condition 1: $\sin x$ is continuous on $[0, \pi]$ ✓
Condition 2: $\sin x$ is differentiable on $(0, \pi)$ ✓
Condition 3:

$$f(0) = \sin 0 = 0$$
$$f(\pi) = \sin \pi = 0$$

So $f(0) = f(\pi)$ ✓
Step 2: Find c

$$f'(x) = \cos x$$
$$f'(c) = 0$$
$$\cos c = 0$$
$$c = \frac{\pi}{2} \quad \text{(in the interval } (0, \pi))$$

Step 3: Verify $c \in (0, \pi)$
$c = \frac{\pi}{2}$ and $0 < \frac{\pi}{2} < \pi$ ✓
Answer: $\boxed{c = \frac{\pi}{2}}$
Geometric Check: The sine curve reaches its peak at $x = \frac{\pi}{2}$, where it has a horizontal tangent. This matches Rolle's prediction! ✓

Example 6: When Rolle's Theorem Fails

Problem: Explain why Rolle's Theorem does not apply to:
(a) $f(x) = |x|$ on $[-1, 1]$
(b) $f(x) = x$ on $[0, 2]$
(c) $f(x) = \frac{1}{x-1}$ on $[0, 2]$

Solution:
(a) $f(x) = |x|$ on $[-1, 1]$
Check conditions:

- Continuous on $[-1, 1]$? Yes ✓
- Differentiable on $(-1, 1)$? No – not differentiable at $x = 0$ (corner point) ×
- $f(-1) = f(1)$? Yes, both equal 1 ✓

Answer: Rolle's Theorem **does not apply** because f is not differentiable on $(-1, 1)$.
Observation: Indeed, there is no point where $f'(x) = 0$ in $(-1, 1)$ since the derivative is either -1 or $+1$ everywhere except at the non-differentiable point $x = 0$.

(b) $f(x) = x$ on $[0, 2]$
Check conditions:

- Continuous on $[0, 2]$? Yes ✓
- Differentiable on $(0, 2)$? Yes ✓
- $f(0) = f(2)$? No – $f(0) = 0$ but $f(2) = 2$ ×

Answer: Rolle's Theorem **does not apply** because $f(0) \neq f(2)$.
Observation: The function $f(x) = x$ has constant derivative $f'(x) = 1$, which is never zero.

(c) $f(x) = \dfrac{1}{x-1}$ on $[0, 2]$
Check conditions:

- Continuous on $[0, 2]$? No – discontinuous at $x = 1$ ×
- (Other conditions irrelevant if continuity fails)

Answer: Rolle's Theorem **does not apply** because f is not continuous on $[0, 2]$ (vertical asymptote at $x = 1$).

Example 7: Application of Rolle's Theorem

Problem: Show that the equation $x^3 + x - 1 = 0$ has exactly one real root.

Solution:
Let $f(x) = x^3 + x - 1$.
Step 1: Show that a root exists (using Intermediate Value Theorem)

$$f(0) = -1 < 0$$
$$f(1) = 1 + 1 - 1 = 1 > 0$$

Since f is continuous and changes sign, by IVT there exists at least one $r \in (0, 1)$ with $f(r) = 0$. ✓
Step 2: Show the root is unique (using Rolle's Theorem by contradiction)
Suppose there are two distinct roots r_1 and r_2 with $r_1 < r_2$.
Then $f(r_1) = 0$ and $f(r_2) = 0$, so $f(r_1) = f(r_2)$.
Since f is a polynomial, it is continuous on $[r_1, r_2]$ and differentiable on (r_1, r_2).
By Rolle's Theorem, there exists $c \in (r_1, r_2)$ such that $f'(c) = 0$.
But:

$$f'(x) = 3x^2 + 1 \geq 1 > 0 \quad \text{for all } x$$

So $f'(x)$ is never zero! This is a contradiction.
Conclusion: The assumption of two distinct roots must be false. Therefore, the equation has **exactly one real root**.
Answer: $\boxed{\text{Exactly one real root exists}}$
Key Technique: Rolle's Theorem can be used to prove uniqueness of roots by contradiction!

Section 4.2 Summary: Rolle's Theorem
Rolle's Theorem:

If f is continuous on $[a,b]$, differentiable on (a,b), and $f(a) = f(b)$, then:

$$\boxed{\exists\, c \in (a,b) : f'(c) = 0}$$

Procedure to Apply Rolle's Theorem:

1. **Verify all three conditions explicitly** (continuity, differentiability, equal endpoints)
2. Find $f'(x)$
3. Solve $f'(c) = 0$
4. Verify that $c \in (a,b)$

Applications:

- Proving uniqueness of roots (by contradiction)
- Foundation for Mean Value Theorem
- Theoretical tool in advanced analysis

⚠ Common Mistakes:

- Not explicitly verifying all three conditions
- Forgetting to check $c \in (a,b)$ after finding it
- Confusing Rolle's Theorem with EVT or MVT
- Trying to apply when $f(a) \neq f(b)$

Exam Tip: Always write "Check conditions:" and list all three. Examiners award marks for this even if you make an error later!

Mean Value Theorem

The Mean Value Theorem (MVT) is one of the most important theorems in calculus. It appears in **every past paper** and forms the theoretical foundation for understanding how derivatives relate to function behavior over intervals.

Theorem 4.3: The Mean Value Theorem (MVT)

If f satisfies **both** conditions:

1. f is **continuous** on the closed interval $[a, b]$
2. f is **differentiable** on the open interval (a, b)

Then there exists at least one number c in (a, b) such that:

$$\boxed{f'(c) = \frac{f(b) - f(a)}{b - a}}$$

Alternative Forms:

Form 1 (Slope form):

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

Form 2 (Change form):

$$f(b) - f(a) = f'(c)(b - a)$$

Form 3 (Mean value form):

$$f(b) = f(a) + f'(c)(b - a)$$

Geometric Interpretation:

There exists at least one point c in (a, b) where the **tangent line is parallel to the secant line** connecting $(a, f(a))$ and $(b, f(b))$.

Physical Interpretation:

If you drive from city A to city B, there must be some instant when your **instantaneous speed** equals your **average speed** for the entire trip.

Proof of the Mean Value Theorem

Proof:

We reduce MVT to Rolle's Theorem using an auxiliary function.

Step 1: Define the auxiliary function

The secant line from $(a, f(a))$ to $(b, f(b))$ has equation:

$$y = f(a) + \frac{f(b) - f(a)}{b - a}(x - a)$$

Define $g(x)$ as the **vertical distance** between $f(x)$ and the secant line:

$$g(x) = f(x) - \left[f(a) + \frac{f(b) - f(a)}{b - a}(x - a)\right]$$

Step 2: Verify Rolle's Theorem applies to g

Continuity: g is continuous on $[a, b]$ since f is continuous. ✓
Differentiability: g is differentiable on (a, b) since f is differentiable. ✓
Equal endpoints:

$$g(a) = f(a) - f(a) - 0 = 0$$
$$g(b) = f(b) - f(a) - \frac{f(b) - f(a)}{b - a}(b - a) = f(b) - f(a) - [f(b) - f(a)] = 0$$

So $g(a) = g(b) = 0$ ✓
Step 3: Apply Rolle's Theorem
By Rolle's Theorem, $\exists\, c \in (a, b)$ such that $g'(c) = 0$.
Step 4: Compute $g'(x)$ and solve

$$g'(x) = f'(x) - \frac{f(b) - f(a)}{b - a}$$

Setting $g'(c) = 0$:

$$f'(c) - \frac{f(b) - f(a)}{b - a} = 0$$
$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

Conclusion: $\exists\, c \in (a, b)$ such that $f'(c) = \dfrac{f(b) - f(a)}{b - a}$. □

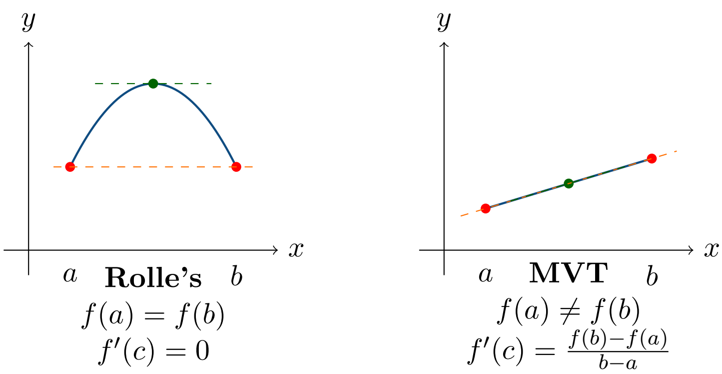
Definition 4.6: Relationship Between Rolle's Theorem and MVT

Rolle's Theorem is a Special Case of MVT:
If $f(a) = f(b)$, then:

$$\frac{f(b) - f(a)}{b - a} = \frac{0}{b - a} = 0$$

So the MVT becomes: $f'(c) = 0$, which is exactly Rolle's Theorem!
In Summary:

- MVT is the **general** statement
- Rolle's Theorem is the **special case** when endpoints are equal
- Both require continuity on $[a, b]$ and differentiability on (a, b)



Example 8: Applying the Mean Value Theorem

Problem: Find the value(s) of c that satisfy the MVT for $f(x) = x^2 - 3x - 1$ on $\left[-\dfrac{1}{7}, \dfrac{13}{7}\right]$.

(From 2024 past paper, Q5 part a)

Solution:

Step 1: Verify MVT conditions

$f(x) = x^2 - 3x - 1$ is a polynomial, so:

- Continuous on $\left[-\frac{1}{7}, \frac{13}{7}\right]$ ✓
- Differentiable on $\left(-\frac{1}{7}, \frac{13}{7}\right)$ ✓

MVT applies!

Step 2: Compute $f(a)$ and $f(b)$

Let $a = -\dfrac{1}{7}$ and $b = \dfrac{13}{7}$:

$$\begin{aligned} f\left(-\frac{1}{7}\right) &= \left(-\frac{1}{7}\right)^2 - 3\left(-\frac{1}{7}\right) - 1 \\ &= \frac{1}{49} + \frac{3}{7} - 1 \\ &= \frac{1 + 21 - 49}{49} \\ &= -\frac{27}{49} \end{aligned}$$

$$\begin{aligned} f\left(\frac{13}{7}\right) &= \left(\frac{13}{7}\right)^2 - 3\left(\frac{13}{7}\right) - 1 \\ &= \frac{169}{49} - \frac{39}{7} - 1 \\ &= \frac{169 - 273 - 49}{49} \\ &= -\frac{153}{49} \end{aligned}$$

Step 3: Compute average rate of change

$$\begin{aligned} \frac{f(b) - f(a)}{b - a} &= \frac{-\frac{153}{49} - \left(-\frac{27}{49}\right)}{\frac{13}{7} - \left(-\frac{1}{7}\right)} \\ &= \frac{-\frac{126}{49}}{\frac{14}{7}} \\ &= \frac{-\frac{126}{49}}{2} \\ &= -\frac{126}{98} \\ &= -\frac{9}{7} \end{aligned}$$

Step 4: Find c by solving $f'(c) = -\dfrac{9}{7}$

$$
\begin{aligned}
f'(x) &= 2x - 3 \\
f'(c) &= -\frac{9}{7} \\
2c - 3 &= -\frac{9}{7} \\
2c &= 3 - \frac{9}{7} \\
2c &= \frac{21-9}{7} \\
2c &= \frac{12}{7} \\
c &= \frac{6}{7}
\end{aligned}
$$

Step 5: Verify $c \in \left(-\frac{1}{7}, \frac{13}{7}\right)$

Since $-\frac{1}{7} < \frac{6}{7} < \frac{13}{7}$ ✓

Answer: $\boxed{c = \frac{6}{7}}$

Example 9: MVT – Simple Application

Problem: Find the value(s) of c satisfying the MVT for $f(x) = x^2 - 1$ on $[0, 2]$. (From 2021 past paper, Q4 – related)

Solution:

Step 1: Verify conditions

Polynomial $\implies$ continuous and differentiable everywhere ✓

Step 2: Compute average rate of change

$$
\begin{aligned}
\frac{f(2) - f(0)}{2 - 0} &= \frac{(4-1) - (0-1)}{2} \\
&= \frac{3 - (-1)}{2} \\
&= \frac{4}{2} \\
&= 2
\end{aligned}
$$

Step 3: Solve $f'(c) = 2$

$$
\begin{aligned}
f'(x) &= 2x \\
f'(c) &= 2 \\
2c &= 2 \\
c &= 1
\end{aligned}
$$

Step 4: Verify $c \in (0, 2)$

$c = 1$ and $0 < 1 < 2$ ✓

Answer: $\boxed{c = 1}$

Interpretation: At $x = 1$, the tangent slope equals the secant slope between $(0, -1)$ and $(2, 3)$.

Example 10: MVT – Trigonometric Function

Problem: Find the value(s) of c satisfying the MVT for $f(x) = \sin x$ on $[0, \pi]$.

Solution:
Step 1: Verify conditions
$\sin x$ is continuous and differentiable everywhere ✓
Step 2: Compute average rate

$$\begin{aligned}\frac{f(\pi) - f(0)}{\pi - 0} &= \frac{\sin \pi - \sin 0}{\pi} \\ &= \frac{0 - 0}{\pi} \\ &= 0\end{aligned}$$

Step 3: Solve $f'(c) = 0$

$$\begin{aligned}f'(x) &= \cos x \\ \cos c &= 0 \\ c &= \frac{\pi}{2} \quad (\text{in } (0, \pi))\end{aligned}$$

Answer: $\boxed{c = \frac{\pi}{2}}$
Note: This is the same as Rolle's Theorem since $f(0) = f(\pi) = 0$!

Example 11: Using MVT to Prove an Inequality

Problem: Use the Mean Value Theorem to prove that:

$$|\sin a - \sin b| \leq |a - b| \quad \text{for all real } a, b$$

(From 2023 past paper, Q4 part b – concept)

Solution:
Step 1: Apply MVT to $f(x) = \sin x$ on $[a, b]$ (assume $a < b$)
$f(x) = \sin x$ is continuous and differentiable everywhere, so MVT applies.
$\exists\, c \in (a, b)$ such that:

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

Step 2: Substitute

$$\begin{aligned}\cos c &= \frac{\sin b - \sin a}{b - a} \\ \sin b - \sin a &= \cos c \cdot (b - a)\end{aligned}$$

Step 3: Take absolute values

$$|\sin b - \sin a| = |\cos c| \cdot |b - a|$$

Step 4: Use $|\cos c| \leq 1$

$$\begin{aligned} |\sin b - \sin a| &\leq 1 \cdot |b - a| \\ &= |b - a| \end{aligned}$$

For $a > b$, the same argument works with roles reversed.

Conclusion:

$$|\sin a - \sin b| \leq |a - b| \quad \text{for all } a, b \in \mathbb{R} \quad \boxed{\text{Q.E.D.}}$$

Interpretation: The sine function cannot change faster than the identity function! This is because $|f'(x)| = |\cos x| \leq 1$ everywhere.

Theorem 4.4: Consequences of the Mean Value Theorem

The MVT has profound theoretical consequences:

Consequence 1 (Constant Function):
If $f'(x) = 0$ for all x in an interval (a, b), then f is constant on (a, b).
Proof: Let $x_1, x_2 \in (a, b)$ with $x_1 < x_2$. By MVT, $\exists\, c \in (x_1, x_2)$ with:

$$f(x_2) - f(x_1) = f'(c)(x_2 - x_1) = 0 \cdot (x_2 - x_1) = 0$$

So $f(x_1) = f(x_2)$ for any two points, hence f is constant. □

Consequence 2 (Functions with Same Derivative):
If $f'(x) = g'(x)$ for all x in (a, b), then f and g differ by a constant on (a, b).
Proof: Let $h(x) = f(x) - g(x)$. Then $h'(x) = f'(x) - g'(x) = 0$ for all x. By Consequence 1, h is constant, so $f(x) - g(x) = C$ for some constant C. □

Consequence 3 (Increasing/Decreasing Test):
If $f'(x) > 0$ for all x in (a, b), then f is strictly increasing on (a, b).
If $f'(x) < 0$ for all x in (a, b), then f is strictly decreasing on (a, b).
Note: This will be explored in detail in Section 4.4!

Section 4.3 Summary: Mean Value Theorem

MVT Statement:
Continuous on $[a, b]$ + differentiable on (a, b) $\implies \exists\, c \in (a, b)$:

$$\boxed{f'(c) = \frac{f(b) - f(a)}{b - a}}$$

Step-by-Step Application:

1. Verify both conditions (continuity and differentiability)
2. Compute $\dfrac{f(b) - f(a)}{b - a}$

3. Find $f'(x)$
4. Solve $f'(c) = \dfrac{f(b) - f(a)}{b - a}$
5. Verify $c \in (a, b)$

Key Facts:

- Rolle's Theorem is MVT with $f(a) = f(b)$
- MVT proves: $f' = 0$ everywhere $\implies$ f is constant
- MVT proves: $f' > 0$ everywhere $\implies$ f is increasing

Exam Alert: MVT appears in **every past paper**! Practice finding c for various function types.

⚠ **Common Mistakes:**

- Not verifying MVT conditions before applying
- Arithmetic errors in computing $\frac{f(b)-f(a)}{b-a}$
- Not checking that $c \in (a, b)$ after finding it
- Confusing MVT with Rolle's Theorem

Increasing and Decreasing Functions

Understanding where a function increases or decreases is essential for curve sketching, optimization, and analyzing function behavior. The first derivative provides a powerful tool for this analysis.

Definition 4.7: Increasing and Decreasing Functions

Let f be defined on an interval I.

Increasing: f is **increasing** on I if:

$$x_1 < x_2 \implies f(x_1) < f(x_2) \quad \text{for all } x_1, x_2 \in I$$

Decreasing: f is **decreasing** on I if:

$$x_1 < x_2 \implies f(x_1) > f(x_2) \quad \text{for all } x_1, x_2 \in I$$

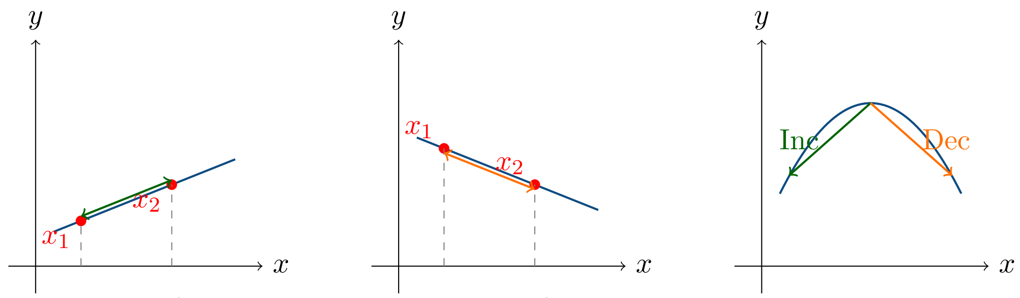
In Words:

- **Increasing:** As you move left to right, the graph goes **up**
- **Decreasing:** As you move left to right, the graph goes **down**

Strictly vs. Non-strictly:

- **Strictly increasing:** $x_1 < x_2 \implies f(x_1) < f(x_2)$ (no equal values)
- **Non-decreasing:** $x_1 < x_2 \implies f(x_1) \leq f(x_2)$ (allows flat parts)

In this course, "increasing" means "strictly increasing" unless otherwise stated.



Increasing $f(x_1) < f(x_2)$ **Decreasing** $f(x_1) > f(x_2)$ **Both** Inc then Dec

Theorem 4.5: Increasing/Decreasing Test (First Derivative Test for Monotonicity)

Let f be continuous on $[a, b]$ and differentiable on (a, b).

(a) If $f'(x) > 0$ for all $x \in (a, b)$, then f is **increasing** on $[a, b]$.
(b) If $f'(x) < 0$ for all $x \in (a, b)$, then f is **decreasing** on $[a, b]$.
(c) If $f'(x) = 0$ for all $x \in (a, b)$, then f is **constant** on $[a, b]$.

Proof Idea:

This follows directly from the Mean Value Theorem! If $x_1 < x_2$ in (a, b), then:

$$f(x_2) - f(x_1) = f'(c)(x_2 - x_1)$$

for some $c \in (x_1, x_2)$.
If $f'(c) > 0$ and $x_2 - x_1 > 0$, then $f(x_2) - f(x_1) > 0$, so $f(x_2) > f(x_1)$ (increasing).
⚠ **Important Subtlety:**
$f'(x) \geq 0$ (allowing $f' = 0$ at isolated points) still implies f is increasing. For example, $f(x) = x^3$ has $f'(0) = 0$, but f is still increasing on $\mathbb{R}$.

Definition 4.8: Sign Chart Method

To find intervals of increase/decrease:
Step 1: Find all critical numbers (where $f'(x) = 0$ or f' undefined)
Step 2: Plot critical numbers on a number line to divide it into intervals
Step 3: Choose a test point in each interval and evaluate $f'(\text{test point})$
Step 4: Create a sign chart:

Interval	Sign of f'	Behavior of f
(a, c_1)	$+$ or $-$	Increasing or Decreasing
(c_1, c_2)	$+$ or $-$	Increasing or Decreasing
$\vdots$	$\vdots$	$\vdots$

Step 5: State intervals of increase and decrease

Example 12: Finding Intervals of Increase and Decrease

Problem: Find the intervals where $f(x) = x^3(4 - x)$ is increasing and decreasing.
(From 2022 past paper, Q5 part a)

Solution:
Step 1: Find $f'(x)$
Expand: $f(x) = 4x^3 - x^4$

$$\begin{aligned} f'(x) &= 12x^2 - 4x^3 \\ &= 4x^2(3 - x) \end{aligned}$$

Step 2: Find critical numbers

$$4x^2(3 - x) = 0 \implies x = 0 \text{ or } x = 3$$

Step 3: Create sign chart
Test points: $x = -1$, $x = 1$, $x = 4$

$$\begin{aligned} f'(-1) &= 4(1)(4) = 16 > 0 \\ f'(1) &= 4(1)(2) = 8 > 0 \\ f'(4) &= 4(16)(-1) = -64 < 0 \end{aligned}$$

	$(-\infty, 0)$	$x = 0$	$(0, 3)$	$x = 3$	$(3, \infty)$
$f'(x)$	$+$	0	$+$	0	$-$
$f(x)$	$\nearrow$	-	$\nearrow$	Max	$\searrow$

Answer:

- **Increasing:** $\boxed{(-\infty, 0) \cup (0, 3) = (-\infty, 3)}$
- **Decreasing:** $\boxed{(3, \infty)}$

Note: Even though $f'(0) = 0$, the function is still increasing through $x = 0$ because f' doesn't change sign there. The point $x = 0$ is an **inflection point**, not a local extremum.

Example 13: Increasing/Decreasing – Another Example

Problem: Find the intervals of increase and decrease for

$$h(x) = -x^3 + 2x^2.$$

(From 2023 past paper, Q3 part b)

Solution:
Step 1: Find the first derivative.

$$\begin{aligned} h'(x) &= -3x^2 + 4x \\ &= x(-3x + 4) \end{aligned}$$

Step 2: Find the critical numbers.

$$\begin{aligned} x(-3x + 4) &= 0 \\ \Rightarrow x &= 0 \quad \text{or} \quad x = \frac{4}{3} \end{aligned}$$

Step 3: Sign chart analysis.
Test points: $x = -1$, $x = 1$, $x = 2$

$$\begin{aligned} h'(-1) &= (-1)(7) = -7 < 0 \\ h'(1) &= (1)(1) = 1 > 0 \\ h'(2) &= (2)(-2) = -4 < 0 \end{aligned}$$

	$(-\infty, 0)$	$x = 0$	$(0, \frac{4}{3})$	$x = \frac{4}{3}$	$(\frac{4}{3}, \infty)$
$h'(x)$	$-$	0	$+$	0	$-$
$h(x)$	$\searrow$	Min	$\nearrow$	Max	$\searrow$

Answer:

- **Increasing:** $\boxed{\left(0, \frac{4}{3}\right)}$
- **Decreasing:** $\boxed{(-\infty, 0) \cup \left(\frac{4}{3}, \infty\right)}$

Bonus: Local minimum at $x = 0$ and local maximum at $x = \frac{4}{3}$.

Example 14: Rational Function

Problem: Find where $f(x) = \dfrac{x^2}{x^2+1}$ is increasing and decreasing.

Solution:
Step 1: Find $f'(x)$ using quotient rule

$$\begin{aligned} f'(x) &= \frac{(x^2+1)(2x) - x^2(2x)}{(x^2+1)^2} \\ &= \frac{2x^3 + 2x - 2x^3}{(x^2+1)^2} \\ &= \frac{2x}{(x^2+1)^2} \end{aligned}$$

Step 2: Find critical numbers
$f'(x) = 0$ when $2x = 0$, so $x = 0$.
$f'(x)$ is never undefined (denominator always positive).
Step 3: Sign chart
Since $(x^2+1)^2 > 0$ always, the sign of $f'(x)$ depends only on $2x$:

	$(-\infty, 0)$	$x = 0$	$(0, \infty)$
$f'(x)$	$-$	0	$+$
$f(x)$	$\searrow$	Min	$\nearrow$

Answer:

- **Increasing:** $\boxed{(0, \infty)}$
- **Decreasing:** $\boxed{(-\infty, 0)}$

Observation: f has a local (and global) minimum at $x = 0$ where $f(0) = 0$.

Section 4.4 Summary

Increasing/Decreasing Test:

Sign of $f'(x)$	Behavior of $f(x)$
$f'(x) > 0$	f is increasing
$f'(x) < 0$	f is decreasing
$f'(x) = 0$	f is constant

Sign Chart Method:

1. Find critical numbers: $f'(x) = 0$ or f' undefined
2. Divide number line into intervals
3. Test sign of f' in each interval
4. Conclude increasing/decreasing based on signs

Exam Alert: This topic appears in almost every past paper! Be comfortable creating sign charts quickly.

Common Mistakes:

- Saying "f is increasing at $x = c$" when you mean "on an interval containing c"
- Including critical numbers in the intervals (use open intervals!)
- Forgetting to check where f' is undefined
- Not simplifying f' before finding critical numbers

Quick Check: If f' changes from $+$ to $-$ at c, what does that tell you about $f(c)$?
Answer: Local maximum!

First Derivative Test

The First Derivative Test uses the sign of the derivative to determine whether critical points are local maxima, local minima, or neither. This is one of the most practical tools for analyzing function behavior.

Theorem 4.6: First Derivative Test

Suppose c is a critical number of a continuous function f.

(a) Local Maximum: If f' changes from **positive to negative** at c, then f has a **local maximum** at c.

(b) Local Minimum: If f' changes from **negative to positive** at c, then f has a **local minimum** at c.

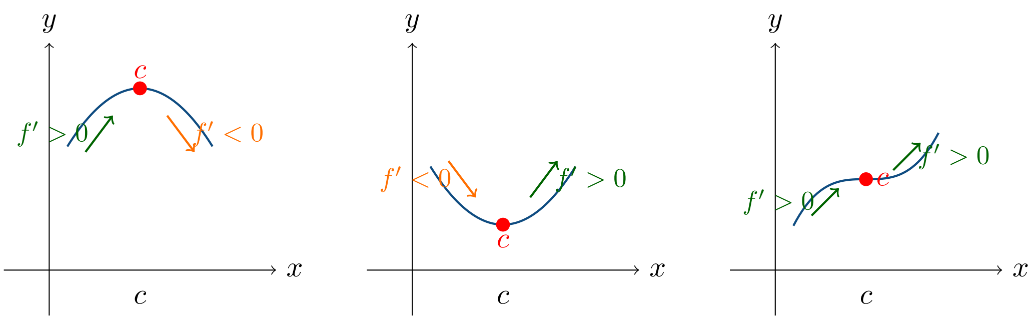
(c) No Extremum: If f' does **not change sign** at c (stays positive or stays negative), then f has **no local extremum** at c.

Intuitive Explanation:

- If f' changes from $+$ to $-$: function goes from increasing to decreasing $\to$ peak (local max)
- If f' changes from $-$ to $+$: function goes from decreasing to increasing $\to$ valley (local min)
- If f' doesn't change sign: function keeps same direction $\to$ no extremum

Important:

The test requires checking the sign of f' on **both sides** of the critical point, not just at the critical point itself!



Local Max $f' : + \to -$ **Local Min** $f' : - \to +$ **No Extremum** $f' : + \to +$

Definition 4.9: Sign Chart for First Derivative Test

To apply the First Derivative Test:

Step 1: Find all critical numbers of f (where $f'(x) = 0$ or f' undefined)

Step 2: Construct a sign chart for $f'(x)$:

- Mark all critical numbers on a number line
- Test the sign of f' in each interval between critical numbers
- Record whether f' is positive or negative in each interval

Step 3: Analyze sign changes at each critical point:

Sign Change of f'	Behavior of f	Classification
$+ \to -$ at c	Increasing $\to$ Decreasing	Local Maximum
$- \to +$ at c	Decreasing $\to$ Increasing	Local Minimum
No sign change	Same direction	No Extremum

Step 4: Evaluate $f(c)$ at each extremum to find the actual maximum/minimum value

Visual Summary:

$$\begin{array}{c|ccc} f' & + & 0 & - \\ \hline f & \nearrow & \text{Max} & \searrow \end{array} \qquad \begin{array}{c|ccc} f' & - & 0 & + \\ \hline f & \searrow & \text{Min} & \nearrow \end{array}$$

Example 15: First Derivative Test – Basic Application

Problem: Find and classify all local extrema of $f(x) = x^3 - 3x + 2$.

Solution:

Step 1: Find critical numbers

$$f'(x) = 3x^2 - 3 = 3(x^2 - 1) = 3(x - 1)(x + 1)$$

Setting $f'(x) = 0$:

$$x = 1 \quad \text{or} \quad x = -1$$

Step 2: Create sign chart for $f'(x) = 3(x - 1)(x + 1)$
Test points: $x = -2$, $x = 0$, $x = 2$

$$\begin{aligned} f'(-2) &= 3(-3)(-1) = 9 > 0 \\ f'(0) &= 3(-1)(1) = -3 < 0 \\ f'(2) &= 3(1)(3) = 9 > 0 \end{aligned}$$

	$(-\infty, -1)$	$x = -1$	$(-1, 1)$	$x = 1$	$(1, \infty)$
$f'(x)$	$+$	0	$-$	0	$+$
$f(x)$	$\nearrow$	**Max**	$\searrow$	**Min**	$\nearrow$

Step 3: Analyze sign changes
At $x = -1$: f' changes from $+$ to $-$ $\implies$ **Local Maximum**
At $x = 1$: f' changes from $-$ to $+$ $\implies$ **Local Minimum**
Step 4: Find the extremum values

$$\begin{aligned} f(-1) &= (-1)^3 - 3(-1) + 2 = -1 + 3 + 2 = 4 \\ f(1) &= (1)^3 - 3(1) + 2 = 1 - 3 + 2 = 0 \end{aligned}$$

Answer:

- **Local Maximum:** $\boxed{f(-1) = 4 \text{ at } x = -1}$
- **Local Minimum:** $\boxed{f(1) = 0 \text{ at } x = 1}$

Example 16: First Derivative Test – No Extremum Case

Problem: Find and classify all local extrema of $g(x) = x^3$.

Solution:
Step 1: Find critical numbers

$$g'(x) = 3x^2$$

Setting $g'(x) = 0$:

$$3x^2 = 0 \implies x = 0$$

Step 2: Sign chart
Test points: $x = -1$, $x = 1$

$$g'(-1) = 3(1) = 3 > 0$$
$$g'(1) = 3(1) = 3 > 0$$

	$(-\infty, 0)$	$x = 0$	$(0, \infty)$
$g'(x)$	$+$	0	$+$
$g(x)$	$\nearrow$	?	$\nearrow$

Step 3: Analyze sign change
At $x = 0$: g' does **not change sign** (stays positive) $\implies$ **No Extremum**
Answer: $\boxed{\text{No local extrema}}$
$x = 0$ is an inflection point, not an extremum.
Key Observation: Even though $g'(0) = 0$, there's no local extremum because the derivative doesn't change sign. The function $g(x) = x^3$ is strictly increasing throughout $\mathbb{R}$.

Example 17: First Derivative Test – Past Paper Style

Problem: For $f(x) = x^4 - 4x^3 + 4x^2$:
(a) Find all critical points
(b) Use the First Derivative Test to classify each
(c) Find the coordinates of all local extrema

Solution:
(a) Find critical points

$$\begin{aligned} f'(x) &= 4x^3 - 12x^2 + 8x \\ &= 4x(x^2 - 3x + 2) \\ &= 4x(x-1)(x-2) \end{aligned}$$

Critical points: $x = 0$, $x = 1$, $x = 2$
(b) First Derivative Test
Test points: $x = -1$, $x = 0.5$, $x = 1.5$, $x = 3$

$$\begin{aligned} f'(-1) &= 4(-1)(-2)(-3) = -24 < 0 \\ f'(0.5) &= 4(0.5)(-0.5)(-1.5) = 1.5 > 0 \\ f'(1.5) &= 4(1.5)(0.5)(-0.5) = -1.5 < 0 \\ f'(3) &= 4(3)(2)(1) = 24 > 0 \end{aligned}$$

	$(-\infty, 0)$	$x = 0$	$(0, 1)$	$x = 1$	$(1, 2)$	$x = 2$	$(2, \infty)$
f'	$-$	0	$+$	0	$-$	0	$+$
f	$\searrow$	**Min**	$\nearrow$	**Max**	$\searrow$	**Min**	$\nearrow$

Classification:

- $x = 0$: f' changes $- \to + \implies$ **Local Minimum**
- $x = 1$: f' changes $+ \to - \implies$ **Local Maximum**
- $x = 2$: f' changes $- \to + \implies$ **Local Minimum**

(c) Find coordinates

$$\begin{aligned} f(0) &= 0 \\ f(1) &= 1 - 4 + 4 = 1 \\ f(2) &= 16 - 32 + 16 = 0 \end{aligned}$$

Answer:

- **Local Minima:** $\boxed{(0, 0) \text{ and } (2, 0)}$
- **Local Maximum:** $\boxed{(1, 1)}$

Example 18: First Derivative Test – Rational Function

Problem: Find and classify all local extrema of $h(x) = \dfrac{x^2 - 1}{x^2 + 1}$.

Solution:
Step 1: Find $h'(x)$ using quotient rule

$$\begin{aligned} h'(x) &= \frac{(x^2 + 1)(2x) - (x^2 - 1)(2x)}{(x^2 + 1)^2} \\ &= \frac{2x^3 + 2x - 2x^3 + 2x}{(x^2 + 1)^2} \\ &= \frac{4x}{(x^2 + 1)^2} \end{aligned}$$

Step 2: Find critical numbers
$h'(x) = 0$ when $4x = 0$, so $x = 0$.
(Denominator never zero, so h' never undefined)
Step 3: Sign chart
Since $(x^2 + 1)^2 > 0$ always, sign depends only on $4x$:

	$(-\infty, 0)$	$x = 0$	$(0, \infty)$
$h'(x)$	$-$	0	$+$
$h(x)$	$\searrow$	**Min**	$\nearrow$

At $x = 0$: h' changes $- \to + \implies$ **Local Minimum**

Step 4: Find minimum value

$$h(0) = \frac{0-1}{0+1} = -1$$

Answer: $\boxed{\text{Local minimum at } (0, -1)}$

Note: This is also a global minimum since $h(x) \to 1$ as $x \to \pm\infty$, and $h(x) \geq -1$ for all x.

Section 4.5 Summary: First Derivative Test

Classification Rules:

Sign Change of f' at c	$f(c)$ is a	Symbol
$+$ to $-$	Local Maximum	$\cap$
$-$ to $+$	Local Minimum	$\cup$
No change	No Extremum	$\nearrow$ or $\searrow$

Procedure:

1. Find all critical numbers ($f'(x) = 0$ or f' undefined)
2. Create sign chart for f'
3. Check sign change at each critical point
4. Evaluate f at extrema to find values

Advantages of First Derivative Test:

- Works even when f'' doesn't exist
- Simultaneously determines intervals of increase/decrease
- More intuitive (based on function behavior)

Common Mistakes:

- Testing f' *at* the critical point instead of on both *sides*
- Forgetting to check critical points where f' is undefined
- Concluding there's an extremum just because $f'(c) = 0$
- Not checking *both* sides of the critical point

Memory Aid: "Up to down = max on top, down to up = min at bottom"

Concavity and Inflection Points

Concavity describes the "direction of bending" of a curve. Understanding concavity helps us sketch curves accurately and provides important information about the behavior of functions.

Definition 4.10: Concavity

Let f be differentiable on an interval I.

Concave Up: f is **concave up** on I if f' is **increasing** on I.

Concave Down: f is **concave down** on I if f' is **decreasing** on I.

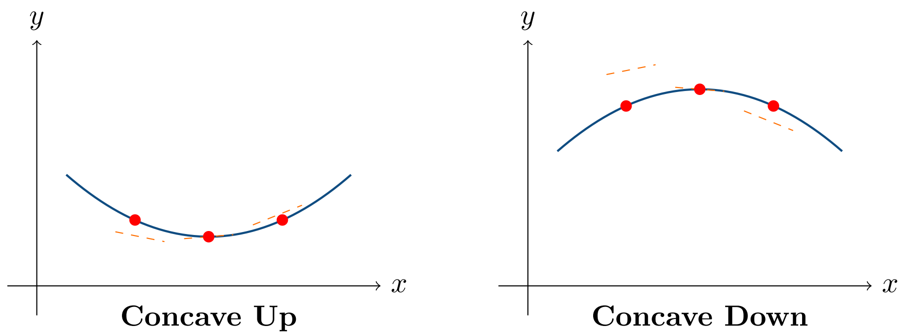
Visual Interpretation:

Concave Up: The graph curves upward, like $\cup$

- Curve "holds water"
- Tangent lines lie **below** the curve
- Slope increases from left to right

Concave Down: The graph curves downward, like $\cap$

- Curve "spills water"
- Tangent lines lie **above** the curve
- Slope decreases from left to right



Concave Up
$f'' > 0$
Tangents below curve

Concave Down
$f'' < 0$
Tangents above curve

Theorem 4.7: Concavity Test (Second Derivative Test for Concavity)

Let f be twice differentiable on an interval I.

(a) If $f''(x) > 0$ for all $x \in I$, then f is **concave up** on I.

(b) If $f''(x) < 0$ for all $x \in I$, then f is **concave down** on I.

Why This Works:

$f''(x) = (f')'(x)$ measures how fast f' is changing.

- If $f'' > 0$, then f' is increasing $\to$ slopes getting steeper $\to$ concave up
- If $f'' < 0$, then f' is decreasing $\to$ slopes getting flatter $\to$ concave down

Quick Summary:

Sign of f''	f' is	Concavity	Shape
$f'' > 0$	Increasing	Concave Up	$\cup$
$f'' < 0$	Decreasing	Concave Down	$\cap$

Definition 4.11: Inflection Point

A point $(c, f(c))$ on the curve $y = f(x)$ is called an **inflection point** if:

- The curve has a tangent line at $(c, f(c))$, AND
- The concavity changes at $x = c$

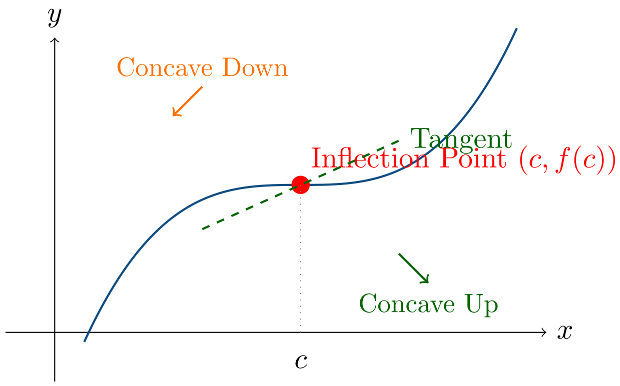
Necessary Condition:
If $(c, f(c))$ is an inflection point, then either:

$$f''(c) = 0 \quad \text{OR} \quad f''(c) \text{ does not exist}$$

Critical Warning:
The converse is **FALSE**! Just because $f''(c) = 0$ doesn't mean c is an inflection point.
Example: $f(x) = x^4$ has $f''(0) = 0$, but $x = 0$ is **not** an inflection point because concavity doesn't change (always concave up).

To Verify an Inflection Point:

1. Find where $f''(x) = 0$ or f'' is undefined
2. Check if f'' changes sign at that point
3. If yes: inflection point. If no: not an inflection point.



Inflection Point: Concavity Changes
From concave down to concave up

Example 19: Finding Concavity and Inflection Points

Problem: For $f(x) = 2x^3 - 3x^2 - 12x$:
(a) Find all inflection points
(b) Determine where f is concave up and concave down
(From 2024 past paper, Q2 part a)

Solution:

(a) Find inflection points
Step 1: Find $f''(x)$

$$\begin{aligned} f'(x) &= 6x^2 - 6x - 12 \\ f''(x) &= 12x - 6 \end{aligned}$$

Step 2: Solve $f''(x) = 0$

$$\begin{aligned} 12x - 6 &= 0 \\ x &= \frac{1}{2} \end{aligned}$$

Step 3: Verify concavity changes
Test points: $x = 0$ (left) and $x = 1$ (right)

$$\begin{aligned} f''(0) &= 12(0) - 6 = -6 < 0 \quad \text{(concave down)} \\ f''(1) &= 12(1) - 6 = 6 > 0 \quad \text{(concave up)} \end{aligned}$$

Concavity changes from down to up at $x = \frac{1}{2}$ ✓
Step 4: Find y-coordinate

$$\begin{aligned} f\left(\frac{1}{2}\right) &= 2\left(\frac{1}{8}\right) - 3\left(\frac{1}{4}\right) - 12\left(\frac{1}{2}\right) \\ &= \frac{1}{4} - \frac{3}{4} - 6 \\ &= -\frac{1}{2} - 6 \\ &= -\frac{13}{2} \end{aligned}$$

Answer (a): $\boxed{\text{Inflection point at } \left(\frac{1}{2}, -\frac{13}{2}\right)}$

(b) Concavity intervals
From the sign chart:

Interval	$f''(x)$	**Concavity**
$(-\infty, \frac{1}{2})$	$-$	Concave Down $\cap$
$(\frac{1}{2}, \infty)$	$+$	Concave Up $\cup$

Answer (b):

- **Concave Up:** $\boxed{\left(\frac{1}{2}, \infty\right)}$
- **Concave Down:** $\boxed{\left(-\infty, \frac{1}{2}\right)}$

Example 20: Concavity – Another Example

Problem: For $h(x) = -x^3 + 2x^2$, find:

(a) Intervals of concavity
(b) All inflection points
(From 2023 past paper, Q3 part b)

Solution:

Step 1: Find $h''(x)$

$$h'(x) = -3x^2 + 4x$$
$$h''(x) = -6x + 4$$

Step 2: Solve $h''(x) = 0$

$$-6x + 4 = 0$$
$$x = \frac{2}{3}$$

Step 3: Sign chart for $h''(x) = -6x + 4$
Test points: $x = 0$ and $x = 1$

$$h''(0) = 4 > 0$$
$$h''(1) = -6 + 4 = -2 < 0$$

Interval	$h''(x)$	Concavity
$(-\infty, \frac{2}{3})$	$+$	Concave Up $\cup$
$(\frac{2}{3}, \infty)$	$-$	Concave Down $\cap$

Answer (a):

- **Concave Up:** $\boxed{\left(-\infty, \frac{2}{3}\right)}$
- **Concave Down:** $\boxed{\left(\frac{2}{3}, \infty\right)}$

Step 4: Find inflection point
Concavity changes at $x = \frac{2}{3}$, so it's an inflection point.
y-coordinate:

$$\begin{aligned} h\left(\frac{2}{3}\right) &= -\left(\frac{2}{3}\right)^3 + 2\left(\frac{2}{3}\right)^2 \\ &= -\frac{8}{27} + 2 \cdot \frac{4}{9} \\ &= -\frac{8}{27} + \frac{8}{9} \\ &= -\frac{8}{27} + \frac{24}{27} \\ &= \frac{16}{27} \end{aligned}$$

Answer (b): $\boxed{\text{Inflection point at } \left(\frac{2}{3}, \frac{16}{27}\right)}$

Example 21: When $f''(c) = 0$ but No Inflection Point

Problem: Show that $f(x) = x^4$ has $f''(0) = 0$, but $x = 0$ is not an inflection point.

Solution:

Step 1: Find $f''(x)$

$$f'(x) = 4x^3$$
$$f''(x) = 12x^2$$

Step 2: Verify $f''(0) = 0$

$$f''(0) = 12(0)^2 = 0 \quad \checkmark$$

Step 3: Check if concavity changes at $x = 0$
Test points: $x = -1$ and $x = 1$

$$f''(-1) = 12(1) = 12 > 0 \quad \text{(concave up)}$$
$$f''(1) = 12(1) = 12 > 0 \quad \text{(concave up)}$$

	$(-\infty, 0)$	$x = 0$	$(0, \infty)$
$f''(x)$	$+$	0	$+$
Concavity	Up	?	Up

Conclusion: Concavity does **not change** at $x = 0$ (stays concave up throughout).
Answer: $\boxed{x = 0 \text{ is NOT an inflection point}}$
⚠ **Key Lesson:** $f''(c) = 0$ is a **necessary** condition for inflection point, but not **sufficient**. Must verify sign change!

Section 4.6 Summary: Concavity and Inflection Points

Concavity Test:

Sign of $f''(x)$	Concavity	Shape
$f'' > 0$	Concave Up	$\cup$
$f'' < 0$	Concave Down	$\cap$

Finding Inflection Points:

1. Find where $f''(x) = 0$ or f'' is undefined
2. Check if f'' changes sign at those points
3. If yes: inflection point. If no: not an inflection point
4. Find y-coordinate: $y = f(c)$

Visual Memory Aid:

- Concave up: "smile" $\cup$ (holds water)
- Concave down: "frown" $\cap$ (spills water)

- Inflection point: where curve changes from smile to frown (or vice versa)

Exam Alert: This topic appears in almost every past paper! Always verify sign change.
⚠ **Common Mistakes:**

- Assuming $f''(c) = 0$ automatically means inflection point
- Not verifying that concavity actually changes
- Confusing inflection points with extrema
- Forgetting to check where f'' is undefined

Quick Test: If you can draw a tangent line that crosses the curve at a point, that point is likely an inflection point!

Second Derivative Test

The Second Derivative Test provides an alternative method for classifying critical points as local maxima or minima. It is often faster than the First Derivative Test when the second derivative is easy to compute.

Theorem 4.8: Second Derivative Test

Suppose f'' is continuous near c and $f'(c) = 0$.

(a) Local Maximum: If $f''(c) < 0$, then f has a **local maximum** at c.

(b) Local Minimum: If $f''(c) > 0$, then f has a **local minimum** at c.

(c) Inconclusive: If $f''(c) = 0$, the test is **inconclusive**. Use the First Derivative Test instead.

Intuitive Explanation:

- If $f''(c) < 0$: curve is concave down at $c \to$ looks like $\cap \to$ peak $\to$ local max
- If $f''(c) > 0$: curve is concave up at $c \to$ looks like $\cup \to$ valley $\to$ local min
- If $f''(c) = 0$: concavity unclear at $c \to$ test fails $\to$ need more analysis

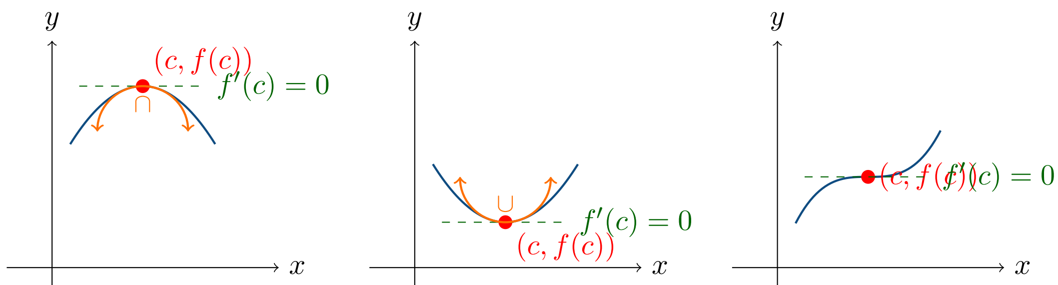
Relationship to Concavity:

The Second Derivative Test essentially checks the concavity at the critical point:

- Concave down ($f'' < 0$) at a critical point $\implies$ local max
- Concave up ($f'' > 0$) at a critical point $\implies$ local min

Critical Requirements:

- Must have $f'(c) = 0$ first! (The test only applies at critical points)
- If $f''(c) = 0$, the test gives **no information**
- If $f''(c)$ doesn't exist, use First Derivative Test



Local Maximum
$f'(c) = 0$
$f''(c) < 0$
(concave down)

Local Minimum
$f'(c) = 0$
$f''(c) > 0$
(concave up)

Inconclusive
$f'(c) = 0$
$f''(c) = 0$
(Use FDT)

Definition 4.12: Comparison of First and Second Derivative Tests

Feature	First Derivative Test	Second Derivative Test
What to check	Sign of f' on both sides of c	Sign of f'' at c
When it works	Always (if f' exists nearby)	Only when $f''(c) \neq 0$
Advantages	Always works; gives inc/dec info	Faster; only check one value
Disadvantages	Requires testing intervals	Fails when $f''(c) = 0$
Use when	f'' hard to compute or $f''(c) = 0$	f'' easy to compute

Decision Guide:

- If f'' is easy to compute and $f''(c) \neq 0 \to$ use Second Derivative Test
- If f'' is complicated or $f''(c) = 0 \to$ use First Derivative Test
- If f'' doesn't exist at $c \to$ use First Derivative Test

Example 22: Second Derivative Test – Basic Application

Problem: Use the Second Derivative Test to find and classify all local extrema of $f(x) = x^3 - 6x^2 + 9x + 1$.

Solution:
Step 1: Find critical numbers

$$\begin{aligned} f'(x) &= 3x^2 - 12x + 9 \\ &= 3(x^2 - 4x + 3) \\ &= 3(x-1)(x-3) \end{aligned}$$

Setting $f'(x) = 0$:

$$x = 1 \quad \text{or} \quad x = 3$$

Step 2: Find $f''(x)$

$$f''(x) = 6x - 12$$

Step 3: Apply Second Derivative Test at each critical point
At $x = 1$:

$$f''(1) = 6(1) - 12 = -6 < 0$$

Since $f''(1) < 0$, f has a **local maximum** at $x = 1$.
At $x = 3$:

$$f''(3) = 6(3) - 12 = 6 > 0$$

Since $f''(3) > 0$, f has a **local minimum** at $x = 3$.
Step 4: Find function values

$$f(1) = 1 - 6 + 9 + 1 = 5$$
$$f(3) = 27 - 54 + 27 + 1 = 1$$

Answer:

- **Local Maximum:** $\boxed{f(1) = 5 \text{ at } x = 1}$
- **Local Minimum:** $\boxed{f(3) = 1 \text{ at } x = 3}$

Example 23: Second Derivative Test – Complete Analysis

Problem: For $f(x) = 2x^3 - 3x^2 - 12x + 1$:
(a) Find all critical points
(b) Use the Second Derivative Test to classify each
(c) Find intervals of concavity
(d) Find inflection points
(From 2024 past paper, Q4 – related)

Solution:
(a) Find critical points

$$\begin{aligned} f'(x) &= 6x^2 - 6x - 12 \\ &= 6(x^2 - x - 2) \\ &= 6(x-2)(x+1) \end{aligned}$$

Critical points: $x = 2$ and $x = -1$
(b) Second Derivative Test
Find $f''(x)$:

$$f''(x) = 12x - 6$$

At $x = -1$:

$$f''(-1) = 12(-1) - 6 = -18 < 0$$

$\implies$ **Local Maximum** at $x = -1$
At $x = 2$:

$$f''(2) = 12(2) - 6 = 18 > 0$$

$\implies$ **Local Minimum** at $x = 2$
Function values:

$$f(-1) = -2 - 3 + 12 + 1 = 8$$
$$f(2) = 16 - 12 - 24 + 1 = -19$$

Answer (a,b):

- Local Maximum: $\boxed{(-1, 8)}$

- Local Minimum: $\boxed{(2, -19)}$

(c) Intervals of concavity

$f''(x) = 12x - 6 = 0 \implies x = \frac{1}{2}$

Interval	$f''(x)$	**Concavity**
$(-\infty, \frac{1}{2})$	$-$	Concave Down
$(\frac{1}{2}, \infty)$	$+$	Concave Up

Answer (c):

- Concave Up: $\boxed{\left(\frac{1}{2}, \infty\right)}$
- Concave Down: $\boxed{\left(-\infty, \frac{1}{2}\right)}$

(d) Inflection point

Concavity changes at $x = \frac{1}{2}$:

$$\begin{aligned} f\left(\frac{1}{2}\right) &= 2\left(\frac{1}{8}\right) - 3\left(\frac{1}{4}\right) - 6 + 1 \\ &= \frac{1}{4} - \frac{3}{4} - 5 \\ &= -\frac{11}{2} \end{aligned}$$

Answer (d): $\boxed{\text{Inflection point at } \left(\frac{1}{2}, -\frac{11}{2}\right)}$

Example 24: When Second Derivative Test Fails

Problem: For $f(x) = x^4$:

(a) Show that $f'(0) = 0$ and $f''(0) = 0$

(b) What does the Second Derivative Test tell us?

(c) Use the First Derivative Test to determine the nature of $x = 0$

Solution:

(a) Compute derivatives

$$\begin{aligned} f'(x) &= 4x^3 \\ f''(x) &= 12x^2 \end{aligned}$$

At $x = 0$:

$$\begin{aligned} f'(0) &= 0 \quad \checkmark \\ f''(0) &= 0 \quad \checkmark \end{aligned}$$

(b) Second Derivative Test result

Since $f'(0) = 0$ and $f''(0) = 0$, the Second Derivative Test is **inconclusive**.

Answer (b): $\boxed{\text{Test fails; gives no information}}$

(c) First Derivative Test

Check sign of $f'(x) = 4x^3$ around $x = 0$:

	$(-\infty, 0)$	$x = 0$	$(0, \infty)$
$f'(x)$	$-$	0	$+$
$f(x)$	$\searrow$	**Min**	$\nearrow$

f' changes from $-$ to $+$ at $x = 0$.

Answer (c): $\boxed{\text{Local (and global) minimum at } x = 0}$

Key Lesson: When Second Derivative Test fails ($f''(c) = 0$), always use First Derivative Test to determine the nature of the critical point.

Example 25: Second Derivative Test – Rational Function

Problem: Use the Second Derivative Test to find and classify all local extrema of $f(x) = x + \dfrac{4}{x}$ on $(0, \infty)$.

Solution:

Step 1: Find critical numbers

$$f'(x) = 1 - \frac{4}{x^2} = \frac{x^2 - 4}{x^2}$$

Setting $f'(x) = 0$:

$$\begin{aligned} x^2 - 4 &= 0 \\ x &= 2 \quad \text{(taking positive root since domain is } (0, \infty)) \end{aligned}$$

Step 2: Find $f''(x)$

$$\begin{aligned} f''(x) &= \frac{d}{dx}\left(1 - \frac{4}{x^2}\right) \\ &= 0 - 4 \cdot \left(-\frac{2}{x^3}\right) \\ &= \frac{8}{x^3} \end{aligned}$$

Step 3: Apply Second Derivative Test

At $x = 2$:

$$f''(2) = \frac{8}{8} = 1 > 0$$

Since $f''(2) > 0$, f has a **local minimum** at $x = 2$.

Step 4: Find function value

$$f(2) = 2 + \frac{4}{2} = 2 + 2 = 4$$

Answer: $\boxed{\text{Local minimum at } (2, 4)}$

Note: This is also a global minimum on $(0, \infty)$ since $f(x) \to \infty$ as $x \to 0^+$ and as $x \to \infty$.

Section 4.7 Summary: Second Derivative Test

Second Derivative Test:

If $f'(c) = 0$, then:

Sign of $f''(c)$	Conclusion
$f''(c) > 0$	Local Minimum at c
$f''(c) < 0$	Local Maximum at c
$f''(c) = 0$	Test Fails (use FDT)

Procedure:

1. Find critical numbers: solve $f'(x) = 0$
2. Compute $f''(x)$
3. Evaluate f'' at each critical number
4. Apply test: $f'' > 0 \to$ min, $f'' < 0 \to$ max, $f'' = 0 \to$ use FDT
5. Find function values at extrema

When to Use Each Test:

- **Use Second Derivative Test:** When f'' is easy to compute
- **Use First Derivative Test:** When f'' is complicated or $f''(c) = 0$

Common Mistakes:

- Applying SDT when $f'(c) \neq 0$ (test only works at critical points!)
- Concluding "no extremum" when $f''(c) = 0$ (test is inconclusive, not negative)
- Not checking that $f'(c) = 0$ first
- Forgetting that SDT can fail

Memory Aid: "Concave up = cup = holds minimum, Concave down = frown = maximum"

Absolute Maximum and Minimum Values

Finding absolute (global) maximum and minimum values is one of the most important practical applications of calculus. These appear in optimization problems across all fields of science, engineering, economics, and everyday life.

Definition 4.13: Review of Absolute Extrema

Recall from Section 4.1:

Absolute Maximum: $f(c)$ is the absolute maximum if $f(c) \geq f(x)$ for all x in the domain.

Absolute Minimum: $f(c)$ is the absolute minimum if $f(c) \leq f(x)$ for all x in the domain.

Key Facts:

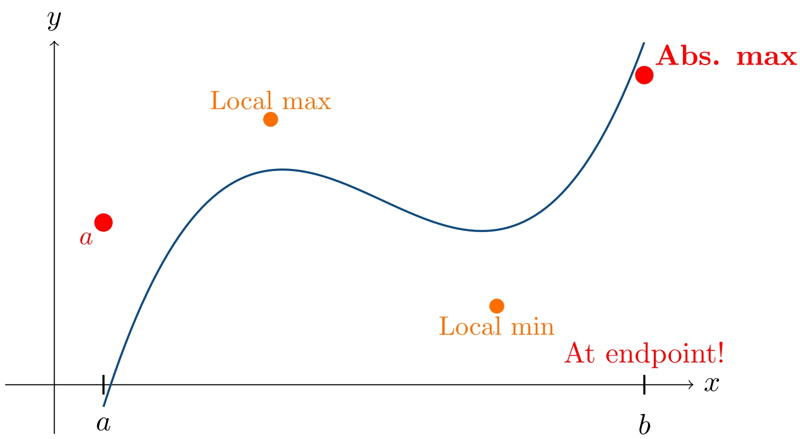
- A function can have at most **one** absolute maximum value (but it may occur at multiple points)
- A function can have at most **one** absolute minimum value
- Absolute extrema can occur at:
  - Critical points inside the domain
  - Endpoints of the domain
  - Points where the function is undefined (boundary of domain)

Important Distinction:

Local extrema: Highest/lowest in a neighbourhood

Absolute extrema: Highest/lowest over the entire domain

Every absolute extremum is a local extremum, but not every local extremum is absolute!



Absolute maximum at endpoint
Local extrema inside interval

Theorem 4.9: Strategy for Finding Absolute Extrema

Case 1: Closed Interval $[a, b]$

By the Extreme Value Theorem, absolute extrema **must exist**. Use the Closed Interval Method:

1. Find all critical numbers in (a, b)
2. Evaluate f at all critical numbers
3. Evaluate f at both endpoints a and b
4. The largest value is the absolute maximum
5. The smallest value is the absolute minimum

Case 2: Open Interval (a, b) **or Unbounded Interval**

Absolute extrema may **not exist**. To find them:

1. Find all critical numbers
2. Evaluate f at critical numbers
3. Check the behavior at endpoints/infinity:
   - $\lim_{x\to a^+} f(x)$, $\lim_{x\to b^-} f(x)$
   - $\lim_{x\to\pm\infty} f(x)$
4. Compare all values

Key Principle:

Absolute extrema can only occur at:

- Critical points (where $f'(x) = 0$ or f' undefined)
- Endpoints of the domain
- Points approaching infinity (as limits)

Example 26: Absolute Extrema on Closed Interval

Problem: Find the absolute maximum and minimum values of $f(x) = x^3 - 3x + 2$ on $[-2, 2]$.

Solution:

Step 1: Find critical numbers in $(-2, 2)$

$$f'(x) = 3x^2 - 3 = 3(x^2 - 1) = 3(x - 1)(x + 1)$$

$f'(x) = 0 \implies x = 1$ or $x = -1$

Both lie in $(-2, 2)$ ✓

Step 2: Evaluate at critical numbers and endpoints

$$\begin{aligned} f(-2) &= -8+6+2=0 \\ f(-1) &= -1+3+2=4 \\ f(1) &= 1-3+2=0 \\ f(2) &= 8-6+2=4 \end{aligned}$$

x	$f(x)$	**Type**
-2	0	Endpoint
-1	4	Critical Number
1	0	Critical Number
2	4	Endpoint

Step 3: Identify extrema
Largest value: 4 (occurs at $x=-1$ and $x=2$)
Smallest value: 0 (occurs at $x=-2$ and $x=1$)
Answer:

- **Absolute Maximum:** $\boxed{f(-1)=f(2)=4}$
- **Absolute Minimum:** $\boxed{f(-2)=f(1)=0}$

Note: The absolute maximum occurs at both a critical point ($x=-1$) and an endpoint ($x=2$)!

Example 27: Absolute Extrema – Past Paper Style

Problem: Find the absolute maximum and minimum of $f(x)=x^4-6x^2+4$ on $[-3,3]$.
(From 2025 past paper, Q1 part iii – related)

Solution:
Step 1: Find critical numbers

$$\begin{aligned} f'(x) &= 4x^3-12x \\ &= 4x(x^2-3) \\ &= 4x(x-\sqrt{3})(x+\sqrt{3}) \end{aligned}$$

Critical numbers: $x=0$, $x=\sqrt{3}$, $x=-\sqrt{3}$
All lie in $(-3,3)$ ✓
Step 2: Evaluate at critical numbers and endpoints

$$\begin{aligned} f(-3) &= 81-54+4=31 \\ f(-\sqrt{3}) &= 9-18+4=-5 \\ f(0) &= 0-0+4=4 \\ f(\sqrt{3}) &= 9-18+4=-5 \\ f(3) &= 81-54+4=31 \end{aligned}$$

x	$f(x)$	**Type**
-3	31	Endpoint
$-\sqrt{3}$	-5	Critical Number
0	4	Critical Number
$\sqrt{3}$	-5	Critical Number
3	31	Endpoint

Answer:

- **Absolute Maximum:** $\boxed{f(-3) = f(3) = 31}$
- **Absolute Minimum:** $\boxed{f(-\sqrt{3}) = f(\sqrt{3}) = -5}$

Example 28: Absolute Extrema on Open Interval

Problem: Find the absolute extrema of $f(x) = \dfrac{x^2}{x^2+1}$ on $(-\infty, \infty)$.

Solution:
Step 1: Find critical numbers

$$\begin{aligned} f'(x) &= \frac{(x^2+1)(2x) - x^2(2x)}{(x^2+1)^2} \\ &= \frac{2x}{(x^2+1)^2} \end{aligned}$$

$f'(x) = 0 \implies x = 0$
Step 2: Evaluate at critical number

$$f(0) = \frac{0}{1} = 0$$

Step 3: Check behavior at infinity

$$\begin{aligned} \lim_{x\to\pm\infty} f(x) &= \lim_{x\to\pm\infty} \frac{x^2}{x^2+1} \\ &= \lim_{x\to\pm\infty} \frac{1}{1+1/x^2} \\ &= 1 \end{aligned}$$

Step 4: Analyze

- $f(0) = 0$ is the only critical value
- $f(x) \to 1$ as $x \to \pm\infty$ (but never reaches 1)
- Since $0 < f(x) < 1$ for all $x \neq 0$, and $f(0) = 0$

Answer:

- **Absolute Minimum:** $\boxed{f(0) = 0}$
- **Absolute Maximum:** $\boxed{\text{None (approaches 1 but never reaches it)}}$

Key Observation: On open or unbounded intervals, absolute extrema may not exist!

Example 29: Absolute Extrema – Trigonometric Function

Problem: Find the absolute extrema of $f(x) = \sin x + \cos x$ on $[0, 2\pi]$.

Solution:
Step 1: Find critical numbers

$$f'(x) = \cos x - \sin x$$

$f'(x) = 0 \implies \cos x = \sin x \implies x = \dfrac{\pi}{4}, \dfrac{5\pi}{4}$
Both in $(0, 2\pi)$ ✓
Step 2: Evaluate at critical numbers and endpoints

$$\begin{aligned} f(0) &= \sin 0 + \cos 0 = 0 + 1 = 1 \\ f\left(\frac{\pi}{4}\right) &= \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} = \frac{2}{\sqrt{2}} = \sqrt{2} \\ f\left(\frac{5\pi}{4}\right) &= -\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} = -\sqrt{2} \\ f(2\pi) &= \sin 2\pi + \cos 2\pi = 0 + 1 = 1 \end{aligned}$$

x	$f(x)$
0	1
$\pi/4$	$\sqrt{2} \approx 1.414$
$5\pi/4$	$-\sqrt{2} \approx -1.414$
2π	1

Answer:

- **Absolute Maximum:** $\boxed{f\left(\frac{\pi}{4}\right) = \sqrt{2}}$
- **Absolute Minimum:** $\boxed{f\left(\frac{5\pi}{4}\right) = -\sqrt{2}}$

Example 30: Absolute Extrema with Endpoints

Problem: A company's profit function is $P(x) = -x^3 + 12x^2 + 60x - 100$ (in thousands of dollars) where x is the number of units produced (in thousands). Production capacity limits x to $[0, 10]$. Find the production level that maximizes profit.

Solution:
Step 1: Find critical numbers in $(0, 10)$

$$\begin{aligned} P'(x) &= -3x^2 + 24x + 60 \\ &= -3(x^2 - 8x - 20) \\ &= -3(x - 10)(x + 2) \end{aligned}$$

$P'(x) = 0 \implies x = 10$ or $x = -2$

Only $x = 10$ is in $[0, 10]$, but it's an endpoint!
So no critical numbers in $(0, 10)$.
Step 2: Evaluate at endpoints

$$P(0) = -100$$
$$P(10) = -1000 + 1200 + 600 - 100 = 700$$

Step 3: Check if there are other critical points
Since $P'(x) = -3(x - 10)(x + 2)$:

- For $x \in (0, 10)$: $P'(x) = -3(\text{negative})(\text{positive}) > 0$
- So P is increasing throughout $(0, 10)$

Answer: Maximum profit occurs at $\boxed{x = 10 \text{ thousand units}}$
Maximum profit: $\boxed{\$700{,}000}$
Business Interpretation: The company should produce at full capacity to maximize profit!

Section 4.8 Summary: Absolute Extrema

Where Absolute Extrema Can Occur:

1. At critical points (where $f'(x) = 0$ or f' undefined)
2. At endpoints of the domain
3. At points approaching boundaries (limits)

Closed Interval Method (for $[a, b]$):

1. Find all critical numbers in (a, b)
2. Evaluate f at: critical numbers + both endpoints
3. Largest value = absolute max; smallest value = absolute min

Open/Unbounded Intervals:

1. Find critical numbers
2. Evaluate f at critical numbers
3. Check limits at boundaries
4. Compare all values
5. Absolute extrema may not exist

Exam Alert: This topic appears in **almost every past paper**! Master the Closed Interval Method.

Most Common Mistakes:

- **Forgetting to evaluate at endpoints** (most common error!)
- Not checking that critical numbers lie inside the interval

- Confusing local extrema with absolute extrema
- On open intervals, claiming an extremum exists when it's only approached

Quick Checklist:

☐ Found all critical numbers?

☐ Checked they're in the interval?

☐ Evaluated at critical numbers?

☐ Evaluated at BOTH endpoints?

☐ Compared ALL values?

Curve Sketching

Curve sketching combines all the techniques we've learned to produce accurate graphs of functions. A systematic approach using derivatives allows us to capture all important features of a function's behavior.

Definition 4.14: Comprehensive Curve Sketching Procedure

To sketch the graph of $y = f(x)$, analyze the following features in order:

Step 1: Domain

- Find all values of x where $f(x)$ is defined
- Identify any excluded values (division by zero, negative under square root, etc.)

Step 2: Intercepts

- y-intercept: Set $x = 0$, find $f(0)$ (if it exists)
- x-intercepts: Set $f(x) = 0$, solve for x

Step 3: Symmetry

- **Even function:** $f(-x) = f(x) \rightarrow$ symmetric about y-axis
- **Odd function:** $f(-x) = -f(x) \rightarrow$ symmetric about origin
- Neither: no special symmetry

Step 4: Asymptotes

- **Vertical:** Check $\lim_{x \to a^{\pm}} f(x) = \pm\infty$ at excluded domain points
- **Horizontal:** Check $\lim_{x \to \pm\infty} f(x) = L$
- **Oblique:** For rational functions where $\deg(p) = \deg(q) + 1$, use long division

Step 5: Intervals of Increase and Decrease

- Find $f'(x)$
- Find critical numbers (where $f'(x) = 0$ or f' undefined)
- Create sign chart for f'
- Determine where $f' > 0$ (increasing) and $f' < 0$ (decreasing)

Step 6: Local Extrema

- Use First or Second Derivative Test
- Identify local maxima and minima
- Find coordinates of extrema

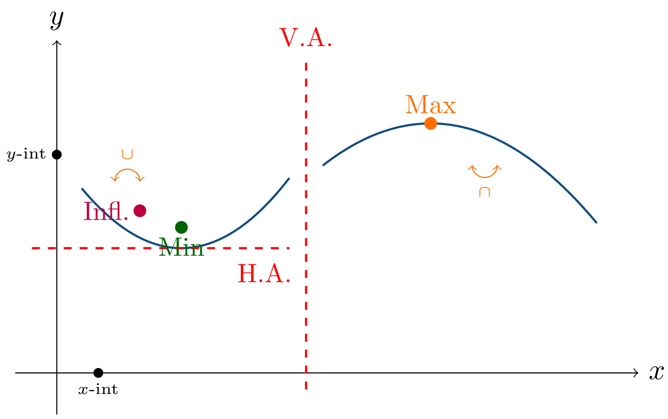
Step 7: Concavity and Inflection Points

- Find $f''(x)$
- Find where $f''(x) = 0$ or f'' undefined

- Create sign chart for f''
- Determine where $f'' > 0$ (concave up) and $f'' < 0$ (concave down)
- Find inflection points (where concavity changes)

Step 8: Sketch the Curve

- Plot intercepts, extrema, and inflection points
- Draw asymptotes as dashed lines
- Connect points smoothly, respecting:
  - Increasing/decreasing behavior
  - Concavity
  - Asymptotic behavior



Complete Curve Sketch
All key features identified

Definition 4.15: Asymptote Quick Reference

Vertical Asymptotes:
The line $x = a$ is a vertical asymptote if:

$$\lim_{x\to a^+} f(x) = \pm\infty \quad \text{or} \quad \lim_{x\to a^-} f(x) = \pm\infty$$

Where to look: Points where denominator $= 0$ (for rational functions)

Horizontal Asymptotes:
The line $y = L$ is a horizontal asymptote if:

$$\lim_{x\to +\infty} f(x) = L \quad \text{or} \quad \lim_{x\to -\infty} f(x) = L$$

For rational functions $\dfrac{p(x)}{q(x)}$:

- $\deg(p) < \deg(q)$: H.A. is $y = 0$
- $\deg(p) = \deg(q)$: H.A. is $y = \dfrac{\text{leading coeff of } p}{\text{leading coeff of } q}$

- $\deg(p) > \deg(q)$: No H.A. (possibly oblique asymptote)

Oblique (Slant) Asymptotes:
If $\deg(p) = \deg(q) + 1$, use polynomial long division:

$$\frac{p(x)}{q(x)} = mx + b + \frac{r(x)}{q(x)}$$

The line $y = mx + b$ is the oblique asymptote.

Example 31: Complete Curve Sketch – Polynomial

Problem: Sketch the graph of $f(x) = 2x^3 - 3x^2 - 12x + 1$ showing all key features. (From 2025 past paper, Q4 part b)

Solution:
Step 1: Domain
Polynomial → domain is all real numbers: $(-\infty, \infty)$
Step 2: Intercepts
y-intercept: $f(0) = 1 \to (0, 1)$
x-intercepts: Solve $2x^3 - 3x^2 - 12x + 1 = 0$ (difficult; use calculator or numerical methods)
Step 3: Symmetry
$f(-x) = -2x^3 - 3x^2 + 12x + 1 \neq f(x)$ and $\neq -f(x)$
No symmetry.
Step 4: Asymptotes
Polynomial → no asymptotes (vertical or horizontal)
As $x \to \infty$, $f(x) \to \infty$
As $x \to -\infty$, $f(x) \to -\infty$
Step 5: Critical points and increasing/decreasing
From previous work (Example 23):

$$f'(x) = 6x^2 - 6x - 12 = 6(x - 2)(x + 1)$$

Critical points: $x = -1$, $x = 2$

Interval	f'	f
$(-\infty, -1)$	$+$	Increasing
$(-1, 2)$	$-$	Decreasing
$(2, \infty)$	$+$	Increasing

Step 6: Local extrema
Local maximum at $x = -1$: $f(-1) = 8 \to (-1, 8)$
Local minimum at $x = 2$: $f(2) = -19 \to (2, -19)$
Step 7: Concavity and inflection points

$$f''(x) = 12x - 6$$

$f''(x) = 0 \implies x = \frac{1}{2}$

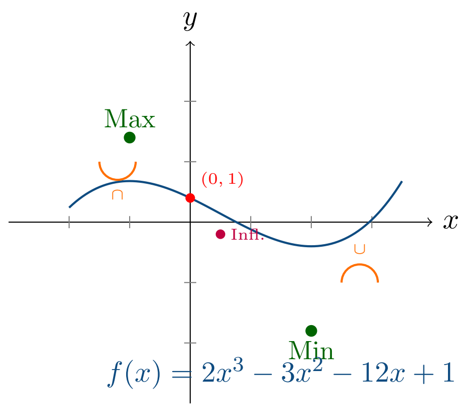
Interval	f''	Concavity
$(-\infty, \frac{1}{2})$	$-$	Concave Down
$(\frac{1}{2}, \infty)$	$+$	Concave Up

Inflection point at $x = \dfrac{1}{2}$: $f\left(\dfrac{1}{2}\right) = -\dfrac{11}{2} \to \left(\dfrac{1}{2}, -\dfrac{11}{2}\right)$

Step 8: Summary and sketch

Key Features:

- y-intercept: $(0, 1)$
- Local max: $(-1, 8)$
- Local min: $(2, -19)$
- Inflection point: $\left(\dfrac{1}{2}, -5.5\right)$
- End behavior: $\lim_{x\to-\infty} f(x) = -\infty$, $\lim_{x\to\infty} f(x) = \infty$



Example 32: Curve Sketch – Rational Function

Problem: Sketch $f(x) = \dfrac{x^2}{x^2 - 4}$ showing all asymptotes, extrema, and concavity.
(From 2022 past paper, Q6 part b)

Solution:

Step 1: Domain
$x^2 - 4 \neq 0 \implies x \neq \pm 2$
Domain: $(-\infty, -2) \cup (-2, 2) \cup (2, \infty)$

Step 2: Intercepts
y-intercept: $f(0) = 0 \to (0, 0)$
x-intercepts: $x^2 = 0 \implies x = 0 \to (0, 0)$

Step 3: Symmetry
$f(-x) = \dfrac{(-x)^2}{(-x)^2 - 4} = \dfrac{x^2}{x^2 - 4} = f(x)$
Even function $\to$ symmetric about y-axis

Step 4: Asymptotes
Vertical: At $x = 2$ and $x = -2$ (where denominator $= 0$)

Check limits:

$$\lim_{x\to 2^+} \frac{x^2}{x^2-4} = +\infty$$
$$\lim_{x\to 2^-} \frac{x^2}{x^2-4} = -\infty$$

(Similar behavior at $x = -2$ by symmetry)
Horizontal: deg(num) = deg(den)

$$\lim_{x\to\pm\infty} \frac{x^2}{x^2-4} = \lim_{x\to\pm\infty} \frac{1}{1-4/x^2} = 1$$

Horizontal asymptote: $y = 1$
Step 5: Increasing/decreasing

$$\begin{aligned} f'(x) &= \frac{(x^2-4)(2x) - x^2(2x)}{(x^2-4)^2} \\ &= \frac{2x^3 - 8x - 2x^3}{(x^2-4)^2} \\ &= \frac{-8x}{(x^2-4)^2} \end{aligned}$$

Critical number: $x = 0$
Since $(x^2-4)^2 > 0$ always, sign depends only on $-8x$:

Interval	f'	f
$(-\infty, -2)$	$+$	Increasing
$(-2, 0)$	$+$	Increasing
$(0, 2)$	$-$	Decreasing
$(2, \infty)$	$-$	Decreasing

Step 6: Local extrema
At $x = 0$: f' changes from $+$ to $-$ $\implies$ Local maximum
$f(0) = 0 \to$ Local max at $(0, 0)$
Step 7: Concavity

$$\begin{aligned} f''(x) &= \frac{-8(x^2-4)^2 + 8x \cdot 2(x^2-4)(2x)}{(x^2-4)^4} \\ &= \frac{-8(x^2-4) + 32x^2}{(x^2-4)^3} \\ &= \frac{24x^2 + 32}{(x^2-4)^3} \end{aligned}$$

Numerator $24x^2 + 32 > 0$ always, so sign depends on $(x^2-4)^3$:

Interval	f''	Concavity
$\lvert x\rvert < 2$	$-$	Concave Down
$\lvert x\rvert > 2$	$+$	Concave Up

No inflection points (concavity changes only at discontinuities)
Step 8: Summary
Key Features:

- Vertical asymptotes: $x = -2$, $x = 2$
- Horizontal asymptote: $y = 1$
- Intercept and local max: $(0, 0)$
- Symmetric about y-axis
- Concave down on $(-2, 2)$
- Concave up on $(-\infty, -2) \cup (2, \infty)$

Section 4.9 Summary: Curve Sketching

8-Step Procedure:

1. Domain
2. Intercepts
3. Symmetry
4. Asymptotes
5. Increasing/Decreasing (f')
6. Local Extrema
7. Concavity (f'')
8. Sketch

Quick Asymptote Rules (Rational Functions):

Condition	Horizontal Asymptote
$\deg(p) < \deg(q)$	$y = 0$
$\deg(p) = \deg(q)$	$y = \dfrac{\text{lead coeff of } p}{\text{lead coeff of } q}$
$\deg(p) > \deg(q)$	None (check oblique)

Exam Alert: Comprehensive curve sketching appears in most past papers. Practice the full procedure!

Common Mistakes:

- Not checking vertical asymptotes at excluded domain points
- Forgetting to verify concavity changes for inflection points
- Drawing curves that violate increasing/decreasing or concavity
- Not using symmetry when present

Pro Tip: Always check your sketch matches all the analysis:

- Does it approach asymptotes correctly?
- Are extrema at the right locations?
- Does concavity match the sign of f''?

Optimization Problems

Optimization problems ask us to find the maximum or minimum value of a quantity subject to given constraints. These are among the most important practical applications of calculus in real-world scenarios.

Definition 4.16: General Strategy for Optimization Problems

Step 1: Understand the Problem

- Read carefully and identify what is being optimized (maximized or minimized)
- Identify the constraints (restrictions on variables)
- Identify known quantities and unknowns

Step 2: Draw a Diagram

- Sketch the physical situation whenever possible
- Label all relevant quantities with variables

Step 3: Introduce Variables

- Let x, y, etc. represent unknown quantities
- Clearly state what each variable represents with units

Step 4: Write the Objective Function

- Express the quantity to be optimized as a function: $Q = f(x, y, \ldots)$
- This is what you want to maximize or minimize

Step 5: Use Constraints to Eliminate Variables

- Use constraint equations to express everything in terms of one variable
- Reduce to: $Q = g(x)$ (function of single variable)

Step 6: Find the Domain

- Determine realistic range for the variable (physical constraints)
- Often $x > 0$ or $a \leq x \leq b$

Step 7: Find Critical Points

- Compute $\dfrac{dQ}{dx}$
- Solve $\dfrac{dQ}{dx} = 0$
- Check endpoints if domain is closed interval

Step 8: Verify Maximum or Minimum

- Use Second Derivative Test: $\dfrac{d^2Q}{dx^2}$
- Or use First Derivative Test (sign chart)

- Or compare values at critical points and endpoints

Step 9: Answer the Question

- State the answer clearly with appropriate units
- Give all requested quantities (dimensions, costs, etc.)
- Check if answer makes physical sense

⚠ Common Pitfalls:

- Substituting numbers too early (before differentiating)
- Not stating what variables represent
- Finding x but not answering what was actually asked
- Forgetting to check endpoints or domain restrictions

Example 33: Maximizing Area with Fixed Perimeter

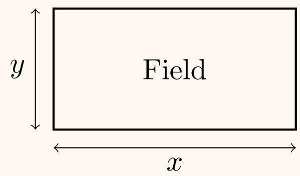
Problem: A farmer has 100 meters of fencing and wants to enclose a rectangular field. What dimensions will maximize the area?

(From 2021 past paper, Q5 part b)

Solution:

Step 1–2: Understand and draw

Let x = length and y = width of rectangle (in meters)



Step 3–4: Variables and objective

Objective: Maximize $A = xy$

Constraint: Perimeter = 100, so $2x + 2y = 100$

Step 5: Eliminate one variable

From constraint: $x + y = 50 \implies y = 50 - x$

Substitute: $A(x) = x(50 - x) = 50x - x^2$

Step 6: Domain

$x > 0$ and $y = 50 - x > 0 \implies 0 < x < 50$

Step 7: Find critical points

$$\frac{dA}{dx} = 50 - 2x$$
$$\frac{dA}{dx} = 0 \implies x = 25$$

Step 8: Verify maximum

$$\frac{d^2A}{dx^2} = -2 < 0 \implies \text{maximum}$$

Step 9: Answer

$$x = 25 \text{ m}$$
$$y = 50 - 25 = 25 \text{ m}$$
$$A_{\max} = 25 \times 25 = 625 \text{ m}^2$$

Answer: $\boxed{\text{Dimensions: } 25 \text{ m} \times 25 \text{ m (a square)}}$

Maximum area: $\boxed{625 \text{ m}^2}$

General Principle: For fixed perimeter, a square maximizes area!

Example 34: Minimizing Surface Area

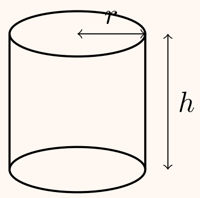
Problem: A cylindrical can must hold 1000 cm³ of liquid. Find the radius and height that minimize the total surface area (including top and bottom).

(From 2023 past paper, Q5 part b – similar)

Solution:

Step 1–2: Setup

Let r = radius (cm), h = height (cm)



Step 3–4: Objective and constraint

Objective: Minimize $S = 2\pi r^2 + 2\pi rh$ (two circles + lateral surface)

Constraint: Volume = 1000, so $\pi r^2 h = 1000$

Step 5: Eliminate h

From constraint: $h = \dfrac{1000}{\pi r^2}$

Substitute:

$$\begin{aligned} S(r) &= 2\pi r^2 + 2\pi r \cdot \frac{1000}{\pi r^2} \\ &= 2\pi r^2 + \frac{2000}{r} \end{aligned}$$

Step 6: Domain

$r > 0$

Step 7: Critical points

$$\frac{dS}{dr} = 4\pi r - \frac{2000}{r^2}$$
$$\frac{dS}{dr} = 0 \implies 4\pi r = \frac{2000}{r^2}$$
$$4\pi r^3 = 2000$$
$$r^3 = \frac{500}{\pi}$$
$$r = \sqrt[3]{\frac{500}{\pi}} \approx 5.42 \text{ cm}$$

Step 8: Verify minimum

$$\frac{d^2S}{dr^2} = 4\pi + \frac{4000}{r^3} > 0 \text{ always} \implies \text{minimum}$$

Step 9: Find h

$$h = \frac{1000}{\pi r^2} = \frac{1000}{\pi\left(\frac{500}{\pi}\right)^{2/3}}$$
$$= \frac{1000}{\pi} \cdot \left(\frac{\pi}{500}\right)^{2/3}$$
$$= 2\sqrt[3]{\frac{500}{\pi}} = 2r \approx 10.84 \text{ cm}$$

Answer: $\boxed{r \approx 5.42 \text{ cm}, \quad h \approx 10.84 \text{ cm}}$

Beautiful Result: Optimal cylinder has $h = 2r$ (height = diameter)!

Example 35: Box with Maximum Volume

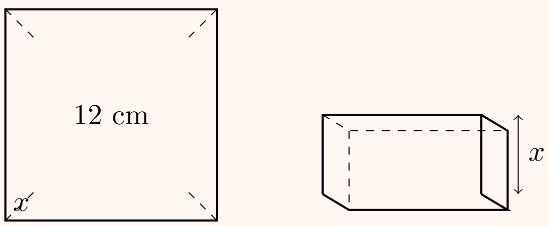
Problem: An open box is made by cutting equal squares from each corner of a 12 cm × 12 cm piece of cardboard and folding up the sides. Find the size of the square cut that maximizes volume.

(From 2024 past paper, Q5 part b – similar)

Solution:

Step 1–2: Diagram

Let x = side length of cut squares (cm)



Step 3–4: Objective

After cutting and folding:

- Length = Width = $12 - 2x$
- Height = x

Objective: Maximize $V = x(12-2x)^2$

Step 5–6: Domain

$x > 0$ and $12 - 2x > 0 \implies 0 < x < 6$

Step 7: Critical points

Expand: $V = x(144 - 48x + 4x^2) = 144x - 48x^2 + 4x^3$

$$\begin{aligned}\frac{dV}{dx} &= 144 - 96x + 12x^2 \\ &= 12(12 - 8x + x^2) \\ &= 12(x-2)(x-6)\end{aligned}$$

$\frac{dV}{dx} = 0 \implies x = 2$ or $x = 6$

Only $x = 2 \in (0,6)$ is valid.

Step 8: Verify maximum

$$\frac{d^2V}{dx^2} = -96 + 24x$$

At $x = 2$: $\frac{d^2V}{dx^2} = -96 + 48 = -48 < 0 \implies$ maximum

Step 9: Answer and volume

$$\begin{aligned}x &= 2 \text{ cm} \\ V_{\max} &= 2(12-4)^2 = 2(64) = 128 \text{ cm}^3\end{aligned}$$

Answer: $\boxed{\text{Cut squares of side 2 cm}}$

Maximum volume: $\boxed{128 \text{ cm}^3}$

Box dimensions: $8 \times 8 \times 2$ cm

Example 36: Closest Point on a Curve

Problem: Find the point on the parabola $y = x^2$ that is closest to the point $(3, 0)$.

Solution:

Step 1–2: Setup

Point on parabola: (x, x^2)

Point given: $(3, 0)$

Distance: $D = \sqrt{(x-3)^2 + (x^2-0)^2}$

Step 3–4: Objective

Minimize D, or equivalently minimize D^2 (easier):

$$f(x) = D^2 = (x-3)^2 + x^4$$

Step 5–6: Domain

$x \in (-\infty, \infty)$

Step 7: Critical points

$$\begin{aligned}f'(x) &= 2(x-3) + 4x^3 \\ &= 4x^3 + 2x - 6\end{aligned}$$

Test $x = 1$: $f'(1) = 4 + 2 - 6 = 0$ ✓
Factor: $f'(x) = 2(2x^3 + x - 3) = 2(x - 1)(2x^2 + 2x + 3)$
The quadratic $2x^2 + 2x + 3$ has no real roots (discriminant ¡ 0).
Only critical point: $x = 1$
Step 8: Verify minimum
$f''(x) = 12x^2 + 2$
$f''(1) = 14 > 0 \implies$ minimum
Step 9: Answer
When $x = 1$: $y = 1^2 = 1$
Answer: $\boxed{\text{Closest point is } (1, 1)}$
Distance: $D = \sqrt{(1 - 3)^2 + 1^2} = \sqrt{5} \approx 2.24$ units

Section 4.10 Summary: Optimization

9-Step Strategy:

1. Understand the problem
2. Draw a diagram
3. Introduce variables
4. Write objective function
5. Use constraints to eliminate variables
6. Find domain
7. Find critical points
8. Verify max/min
9. Answer the question (with units!)

Classic Results to Remember:

- Fixed perimeter → max area: **square**
- Fixed volume cylinder → min surface area: $h = 2r$
- Sum of two numbers fixed → max product: **equal numbers**

Exam Alert: Optimization appears frequently in past papers. Always show complete setup and working!

Top Mistakes in Optimization:

- Not using constraint to reduce to one variable
- Substituting numbers before differentiating
- Finding x but not answering what was asked
- Forgetting units in final answer
- Not checking domain restrictions
- Not verifying whether it's a max or min

Success Checklist:

☐ Drew a diagram?

☐ Stated what variables represent?

☐ Used constraint correctly?

☐ Verified max/min?

☐ Answered actual question with units?

☐ Does answer make physical sense?

Practice Exercises

Problem 1: Finding Critical Numbers

Find all critical numbers for each function:

(a) $f(x) = 3x^4 - 4x^3 - 12x^2 + 5$
(b) $g(x) = x^5 - 5x^4 + 5x^3$
(c) $h(x) = x\sqrt{x^2+1}$
(d) $k(x) = \dfrac{x^2+1}{x^2-1}$
(e) $p(x) = x^{2/3}(x-5)$
(f) $q(x) = 2\sin x + \cos(2x)$ on $[0, 2\pi]$

Remember: Critical numbers occur where $f'(c) = 0$ OR $f'(c)$ does not exist.

Problem 2: Extreme Value Theorem – Theory

(a) State the Extreme Value Theorem precisely, including all conditions.

(b) Give an example of a function on $(0, 1)$ that has no absolute minimum. Explain why EVT does not apply.

(c) Give an example of a function on $[0, 1]$ that is not continuous and has no absolute maximum. Explain.

(d) Does $f(x) = x^3 - 3x$ satisfy the conditions of EVT on $[-2, 2]$? If so, guarantee the existence of absolute extrema.

(e) Explain why $f(x) = \tan x$ on $[0, \pi]$ does not satisfy EVT conditions.

Problem 3: Closed Interval Method

Use the Closed Interval Method to find the absolute maximum and minimum values:

(a) $f(x) = x^3 - 6x^2 + 9x + 1$ on $[0, 4]$
(b) $g(x) = x^4 - 2x^2 + 3$ on $[-2, 2]$
(c) $h(x) = 2x - 3x^{2/3}$ on $[-1, 3]$
(d) $k(x) = x + \dfrac{9}{x}$ on $\left[\dfrac{1}{2}, 4\right]$
(e) $f(x) = x\sqrt{4 - x^2}$ on $[-2, 2]$
(f) $g(x) = e^x - 2e^{-x}$ on $[0, 2]$

Problem 4: Verifying Rolle's Theorem

For each function, verify all three conditions of Rolle's Theorem on the given interval, then find all values of c guaranteed by the theorem:

(a) $f(x) = x^3 - 3x^2 + 2x$ on $[0, 2]$
(b) $g(x) = x^2(1 - x)$ on $[0, 1]$
(c) $h(x) = \sin(2x)$ on $[0, \pi]$
(d) $k(x) = e^x(x - 1)$ on $[0, 1]$
(e) $f(x) = x^4 - 2x^2$ on $[-\sqrt{2}, \sqrt{2}]$

Always state all three conditions explicitly before finding c.

Problem 5: Rolle's Theorem – Applications

(a) Show that the equation $x^5 + 2x^3 + x - 1 = 0$ has exactly one real root.
Hint: Use IVT to show existence, then Rolle's Theorem to show uniqueness.
(b) Prove that between any two real roots of $e^x \sin x = 0$, there is at least one root of $e^x(\sin x + \cos x) = 0$.
(c) If f is differentiable on $\mathbb{R}$, $f(1) = f(3) = f(5) = 0$, prove that $f'(c) = 0$ for at least two values of c in $(1, 5)$.
(d) Can Rolle's Theorem be applied to $f(x) = 1 - |x - 1|$ on $[0, 2]$? Explain why or why not.
(e) Let $f(x) = x^3 - x$. Find all c in $(-1, 1)$ where $f'(c) = 0$, and verify this is consistent with Rolle's Theorem.

Problem 6: Applying MVT – Finding c

For each function and interval, verify MVT conditions then find all values of c:
(a) $f(x) = x^3 - 3x + 1$ on $[-2, 2]$
(b) $f(x) = \sqrt{x}$ on $[0, 4]$
(c) $f(x) = \dfrac{1}{x}$ on $[1, 3]$
(d) $f(x) = x^2 + \dfrac{1}{x}$ on $\left[\dfrac{1}{2}, 2\right]$
(e) $f(x) = x - 2\sin x$ on $[0, \pi]$

Steps: verify conditions → compute $\frac{f(b)-f(a)}{b-a}$ → solve $f'(c) =$ that value → verify $c \in (a, b)$

Problem 7: MVT – Past Paper Style

(a) Verify that $f(x) = x^2 - 3x - 1$ satisfies the conditions of the Mean Value Theorem on $\left[-\dfrac{1}{7}, \dfrac{13}{7}\right]$. Find the value(s) of c guaranteed by the theorem.
(From 2024 past paper, Q5 part a)
(b) For $f(x) = x^3 - 5x^2 - 3x$ on $[1, 3]$, verify MVT and find all c such that:

$$f'(c) = \frac{f(3) - f(1)}{3 - 1}$$

(From 2023 past paper, Q4 part a – modified)
(c) Apply the Mean Value Theorem to $f(x) = x^2 - 1$ on $[0, 2]$ and interpret the result geometrically.
(From 2021 past paper, Q4 – related)
(d) Find all values of c satisfying MVT for $f(x) = \ln x$ on $[1, e]$.

Problem 8: MVT – Proofs and Applications

(a) State the Mean Value Theorem precisely, including all necessary conditions.
(From 2022 past paper, Q1 part xii)
(b) Use the MVT to prove that $|\cos a - \cos b| \leq |a - b|$ for all real numbers a and b.
(c) Use the MVT to prove: If $f'(x) = g'(x)$ for all x in (a, b), then f and g differ by a constant on (a, b).

(d) A car travels 100 km in 50 minutes. Use the MVT to prove that at some moment during the trip, the speedometer read exactly 120 km/h.

(e) Suppose f is differentiable on $[a, b]$ and $|f'(x)| \leq M$ for all $x \in (a, b)$. Prove that:

$$|f(b) - f(a)| \leq M|b - a|$$

Problem 9: MVT – Comprehensive Problems

(a) For $f(x) = x(x-1)(x-2)$ on $[0, 2]$, verify that MVT applies. Find all values of c in $(0, 2)$ where the tangent is parallel to the secant.

(b) Does MVT apply to $f(x) = |x - 1|$ on $[0, 2]$? Explain.

(c) Let $f(x) = x^4$. Find all c in $(0, 2)$ such that $f'(c) = \dfrac{f(2) - f(0)}{2}$.

(d) Verify MVT for $f(x) = \sin x$ on $\left[0, \dfrac{\pi}{2}\right]$ and find c.

(e) A function f satisfies $f(1) = 5$ and $f'(x) \geq 2$ for $1 \leq x \leq 4$. What is the smallest possible value of $f(4)$? Use MVT to justify.

Problem 10: Intervals of Increase and Decrease

Find the intervals where each function is increasing and decreasing:

(a) $f(x) = x^3(4 - x)$

(From 2022 past paper, Q5 part a)

(b) $h(x) = -x^3 + 2x^2$

(From 2023 past paper, Q3 part b)

(c) $f(x) = 2x^3 - 3x^2 - 12x + 1$

(From 2024 past paper Q5 – related)

(d) $g(x) = x^4 - 4x^3 + 4x^2$

(e) $f(x) = xe^{-x}$

(f) $h(x) = \dfrac{x}{x^2 + 1}$

Problem 11: Increasing/Decreasing – Various Functions

Determine where the following functions are increasing and decreasing:

(a) $f(x) = x^2 \ln x$ on $(0, \infty)$

(b) $g(x) = \dfrac{x^2 - 4}{x}$ on its domain

(c) $h(x) = x + \dfrac{16}{x}$ on $(0, \infty)$

(d) $k(x) = \sin x + \cos x$ on $[0, 2\pi]$

(e) $f(x) = x - 2 \arctan x$ on $\mathbb{R}$

(f) $g(x) = (x^2 - 1)e^{-x}$ on $\mathbb{R}$

Problem 12: Sign Charts and Monotonicity

For each function:

- Find all critical numbers
- Construct a complete sign chart for $f'(x)$

- State intervals where f is increasing and decreasing
- Identify any local extrema

(a) $f(x) = x^4 - 8x^2 + 16$
(b) $g(x) = x^2 e^{-x}$
(c) $h(x) = \dfrac{x^2}{x-1}$
(d) $k(x) = x^3 - 6x^2 + 9x + 1$

Problem 13: First Derivative Test – Basic

Use the First Derivative Test to find and classify all local extrema:
(a) $f(x) = x^3 - 12x + 5$
(b) $g(x) = x^4 - 4x^3 + 6$
(c) $h(x) = \dfrac{x^2+1}{x^2-1}$
(d) $k(x) = x^2 e^{-x}$
(e) $f(x) = \sin^2 x - \cos x$ on $[0, 2\pi]$
(f) $g(x) = x + 2\sin x$ on $[0, 2\pi]$

Problem 14: First Derivative Test – Past Papers

(a) Find all local maximum and minimum values of $f(x) = x^3(4-x)$. State the coordinates of each extremum.
(From 2022 past paper, Q5 part a)
(b) For $h(x) = -x^3 + 2x^2$, find:

- All critical points
- Local maximum and minimum using the first derivative test
- Coordinates of all turning points

(From 2023 past paper, Q3 part b)
(c) Find and classify all local extrema of $f(x) = x^4 - 4x^3 + 4x^2$.
(d) For $f(x) = 2x^3 - 9x^2 + 12x - 3$, find all local maxima and minima.

Problem 15: First Derivative Test – Mixed

(a) For $f(x) = x + \sin x$ on $[0, 2\pi]$, use the First Derivative Test to determine if there are any local extrema.
(b) Show that $g(x) = x^5 + x^3 + x$ has no local extrema.
(c) Find and classify the local extrema of $h(x) = x^{2/3}(x-4)$.
(d) For $k(x) = |x^2 - 4|$, identify all critical numbers and classify them.
(e) Determine all local extrema of $f(x) = \dfrac{x^3}{x^2+1}$.

Problem 16: Concavity and Inflection Points

For each function, find:

- Intervals where the function is concave up and concave down
- All inflection points (with coordinates)

(a) $f(x) = x^3 - 9x^2 + 24x - 10$
(b) $g(x) = 2x^3 - 3x^2 - 12x$
(From 2024 past paper, Q2 part a)
(c) $h(x) = -x^3 + 2x^2$
(From 2023 past paper, Q3 part b)
(d) $k(x) = x^4 - 6x^2 + 8$
(e) $f(x) = xe^{-x}$
(f) $g(x) = \ln(x^2 + 4)$

Problem 17: Concavity – Theory and Practice

(a) Explain the difference between an inflection point and a critical point.
(b) Give an example where $f''(c) = 0$ but c is NOT an inflection point.
(c) For $f(x) = x^4 - 4x^3$, find all inflection points and intervals of concavity.
(d) Show that $g(x) = \sin x$ has inflection points at $x = n\pi$ for all integers n.
(e) Find the inflection point(s) of $h(x) = \dfrac{x}{x^2 + 1}$ and verify that concavity actually changes.
(f) For what values of a and b does $f(x) = ax^3 + bx^2$ have an inflection point at $(1, 2)$?

Problem 18: Concavity – Comprehensive

(a) For $f(x) = x^3 - 3x + 2$:

- Find intervals of concavity
- Find inflection point(s)
- Sketch the curve showing concavity

(b) Determine concavity and inflection points for $g(x) = x^2 e^{-x}$.
(c) For $h(x) = \dfrac{x^2 - 1}{x^2 + 1}$:

- Find where $h''(x) = 0$
- Determine intervals of concavity
- Find all inflection points

(d) Show that $f(x) = x^4 + x^2$ is concave up on $(-\infty, \infty)$ and has no inflection points.

Problem 19: Second Derivative Test – Basic

Use the Second Derivative Test to find and classify all local extrema:

(a) $f(x) = x^3 - 12x + 2$
(b) $g(x) = 2x^3 - 9x^2 + 12x + 1$
(c) $h(x) = x^4 - 4x^3 + 10$
(d) $k(x) = x - 2\ln x$ on $(0, \infty)$
(e) $f(x) = xe^{-x^2}$
(f) $g(x) = x + \dfrac{4}{x}$ on $(0, \infty)$

Problem 20: Second Derivative Test – Past Papers

(a) For $f(x) = 2x^3 - 3x^2 - 12x + 1$:

- Find all critical points
- Use the Second Derivative Test to classify each
- Find inflection points
- Determine intervals of concavity

(From 2024 past paper, Q4 – related)
(b) Apply the Second Derivative Test to $g(x) = x^3 - 6x^2 + 9x + 5$.
(c) For $h(x) = x^4 - 2x^2 + 3$, use SDT to classify all local extrema.
(d) Find and classify all critical points of $k(x) = x^2e^{-x}$ using the Second Derivative Test.

Problem 21: When Second Derivative Test Fails

(a) For $f(x) = x^3$, show that the Second Derivative Test is inconclusive at $x = 0$. Use the First Derivative Test to determine the nature of this critical point.

(b) Consider $g(x) = x^4$. Verify that $g'(0) = 0$ and $g''(0) = 0$. What does the Second Derivative Test tell you? Use another method to classify $x = 0$.

(c) For $h(x) = x^5$, explain why the Second Derivative Test fails at $x = 0$. What is the correct classification?

(d) Find all critical points of $k(x) = x^4 - 4x^2$. For which critical points does the Second Derivative Test work? For which does it fail?

(e) State precisely when the Second Derivative Test is inconclusive, and what should be done in such cases.

Problem 22: Absolute Extrema – Closed Intervals

Find the absolute maximum and minimum values on the given interval:

(a) $f(x) = 2x^3 - 3x^2 - 12x + 1$ on $[-2, 3]$
(From 2025 past paper, Q4)
(b) $f(x) = x^4 - 6x^2 + 4$ on $[-3, 3]$
(From 2025 past paper, Q1 part iii – related)
(c) $g(x) = x^3(4 - x)$ on $[-1, 5]$
(d) $h(x) = \sin x + \cos x$ on $[0, 2\pi]$
(e) $k(x) = x - 2\sin x$ on $[0, 2\pi]$

(f) $f(x) = x^2 e^{-x}$ on $[0, 4]$

Problem 23: Absolute Extrema – Various Domains

(a) Find the absolute extrema of $f(x) = \dfrac{x^2}{x^2+1}$ on $[-2, 2]$.

(b) For $g(x) = x\sqrt{4-x^2}$ on $[-2, 2]$, find absolute max and min.

(c) Find absolute extrema of $h(x) = x^{2/3}(6-x)$ on $[-1, 8]$.

(d) For $k(x) = \ln(x^2+x+1)$ on $[-1, 2]$, determine absolute extrema.

(e) Find absolute max and min of $f(x) = |x^2 - 4|$ on $[-3, 3]$.

(f) For $g(x) = x + \dfrac{1}{x}$ on $\left[\dfrac{1}{2}, 3\right]$, find absolute extrema.

Problem 24: Absolute Extrema – Open Intervals

(a) Does $f(x) = \dfrac{x^2-1}{x^2+1}$ have an absolute maximum on $(-\infty, \infty)$? If so, find it.

(b) Find the absolute minimum of $g(x) = x^2 + \dfrac{1}{x^2}$ on $(0, \infty)$.

(c) Does $h(x) = xe^{-x}$ have an absolute maximum on $(0, \infty)$? Justify.

(d) For $k(x) = x - \ln x$ on $(0, \infty)$, find the absolute minimum.

(e) Determine if $f(x) = \arctan x$ has absolute extrema on $(-\infty, \infty)$.

Problem 25: Complete Curve Sketching – Polynomials

For each function, provide a complete analysis including:

- Domain and intercepts
- Symmetry
- Asymptotes (if any)
- Intervals of increase/decrease
- Local extrema
- Intervals of concavity
- Inflection points
- Sketch the curve

(a) $f(x) = 2x^3 - 3x^2 - 12x + 1$
(From 2025 past paper, Q4 part b)
(b) $f(x) = x^4 - 4x^3 + 4x^2$
(c) $h(x) = -x^3 + 2x^2$
(From 2023 past paper, Q3 part b)
(d) $f(x) = x^3 - 6x^2 + 9x + 1$

Problem 26: Curve Sketching – Rational Functions

For each rational function, provide complete analysis and sketch:

(a) $f(x) = \dfrac{x^2}{x^2-4}$
(From 2022 past paper, Q6 part b)
(b) $g(x) = \dfrac{x}{x^2+1}$
(c) $h(x) = \dfrac{x^2-1}{x}$
Hint: Use polynomial long division to find oblique asymptote.
(d) $k(x) = \dfrac{x^2+1}{x^2-1}$
(e) $f(x) = \dfrac{2x}{x^2+1}$

Problem 27: Curve Sketching – Mixed Functions

Provide complete curve analysis and sketch for:

(a) $f(x) = xe^{-x}$
(b) $g(x) = x^2 \ln x$ on $(0, \infty)$
(c) $h(x) = \dfrac{e^x}{1+e^x}$
(d) $k(x) = x - 2\arctan x$
(e) $f(x) = x\sqrt{4-x^2}$ on $[-2, 2]$

Problem 28: Classical Optimization

(a) A farmer has 240 metres of fencing. She wants to enclose a rectangular field and divide it into two equal parts with a fence parallel to one side. What dimensions maximize the total enclosed area?

(b) An open box is made from a 16 cm × 16 cm piece of cardboard by cutting equal squares from each corner and folding up the sides. Find the size of the squares cut that maximizes the volume.

(From 2024 past paper, Q5 part b – similar)

(c) A cylindrical can (with lid) must hold exactly 500π cm³. Find the radius and height that minimize the total surface area.

(d) A rectangular poster must have 100 cm² of printed area with margins of 2 cm on top and bottom and 1 cm on each side. What dimensions minimize the total poster area?

(e) A rectangle is inscribed in a semicircle of radius r with one side on the diameter. Find the dimensions that maximize the rectangle's area.

Problem 29: Optimization – Advanced

(a) A window consists of a rectangle surmounted by a semicircle. The perimeter of the window is 10 metres. Find the dimensions that maximize the area.

(From 2023 past paper, Q5 part b – similar)

(b) Find two positive numbers whose sum is 50 and whose product is as large as possible.

(c) Find two positive numbers whose product is 100 and whose sum is as small as possible.

(d) A piece of wire 10 cm long is cut into two pieces. One piece is bent into a circle, the other into a square. How should the wire be cut to:

- Minimize the total area?
- Maximize the total area?

(e) A right circular cone has volume V. Find the ratio of height to base radius that minimizes the surface area.

Problem 30: Optimization – Applications

(a) A company produces x units at cost $C(x) = 500 + 20x + 0.1x^2$ dollars. The revenue from selling x units is $R(x) = 100x - 0.2x^2$ dollars. Find the production level that maximizes profit.

(b) A printed page is to have total area 80 cm^2 with 1 cm margins on top and bottom and 2 cm margins on each side. What dimensions maximize the printed area?

(c) Find the point on the line $y = 2x + 1$ that is closest to the origin.

(d) A Norman window has the shape of a rectangle surmounted by a semicircle. If the perimeter is fixed at 30 feet, find the dimensions that admit the most light.

(e) A closed cylindrical can is to be made with 300π cm^2 of material (including top and bottom). Find the dimensions that maximize the volume.

(f) A track is to be built in the shape of a rectangle with semicircles at each end. The inside of the track must enclose an area of 10,000 m^2. Find the dimensions that minimize the perimeter.

Topic Frequency in Exams (2021–2025):

Topic	Problems	Exam Frequency
Mean Value Theorem	6–9	**Every Exam**
Absolute Max/Min	22–24	**Every Exam**
Increasing/Decreasing	10–12	**Every Exam**
Concavity & Inflection	16–18	**Very Frequent**
Second Derivative Test	19–21	**Very Frequent**
First Derivative Test	13–15	High
Curve Sketching	25–27	High
Optimization	28–30	High
Rolle's Theorem	4–5	Medium
EVT & Critical Numbers	1–3	Medium

Chapter 5

Integral Calculus and Applications

Antiderivatives and Indefinite Integrals

Integration is the reverse process of differentiation. While differentiation asks "what is the rate of change?", integration asks "what function has this rate of change?" This fundamental concept forms the basis of integral calculus.

Definition 5.1: Antiderivative

A function F is called an **antiderivative** of f on an interval I if:

$$\boxed{F'(x) = f(x) \quad \text{for all } x \in I}$$

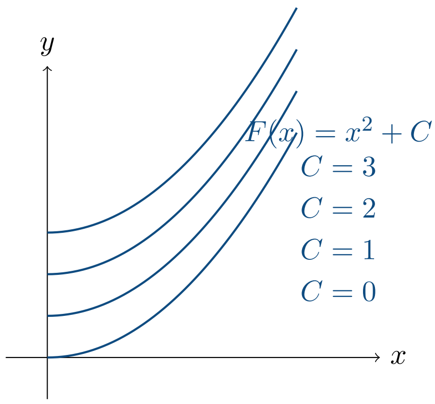
In Words:
An antiderivative of f is any function whose derivative equals f.

Key Examples:

- If $f(x) = 2x$, then $F(x) = x^2$ is an antiderivative because $F'(x) = 2x$
- If $f(x) = \cos x$, then $F(x) = \sin x$ is an antiderivative because $F'(x) = \cos x$
- If $f(x) = e^x$, then $F(x) = e^x$ is an antiderivative because $F'(x) = e^x$

Important Fact:
Antiderivatives are **not unique**! If F is an antiderivative of f, then so is $F + C$ for any constant C, because:

$$\frac{d}{dx}[F(x) + C] = F'(x) + 0 = f(x)$$



Multiple Antiderivatives
All have derivative $f(x) = 2x$

Theorem 5.1: General Form of Antiderivatives

If F is an antiderivative of f on an interval I, then the **most general antiderivative** of f on I is:

$$\boxed{F(x) + C}$$

where C is an arbitrary constant, called the **constant of integration**.

Why This Works:

If G is any other antiderivative of f, then:

$$G'(x) = f(x) = F'(x)$$

This means $(G - F)' = 0$, so $G - F$ must be constant. Therefore:

$$G(x) = F(x) + C$$

for some constant C.

Conclusion:

Any two antiderivatives of the same function differ only by a constant!

Definition 5.2: Indefinite Integral

The **indefinite integral** of f with respect to x is defined as:

$$\boxed{\int f(x)\,dx = F(x) + C}$$

where F is any antiderivative of f and C is an arbitrary constant.

Notation and Terminology:

$$\int f(x)\,dx = F(x) + C$$

$\int$ ↓ Integral; $f(x)$ ↓ Integrand; dx ↓ Differential; $F(x)$ ↓ Antiderivative; C ↓ Constant of integration

Reading: "The integral of $f(x)$ with respect to x equals $F(x)$ plus C"

Relationship to Differentiation:

Integration and differentiation are **inverse operations**:

$$\frac{d}{dx}\left[\int f(x)\,dx\right] = f(x)$$

$$\int f'(x)\,dx = f(x) + C$$

⚠ Always Include $+C$!

The constant of integration is crucial for indefinite integrals. Forgetting it is one of the most common mistakes in integration.

Example 1: Finding Antiderivatives

Problem: Find the general antiderivative (indefinite integral) of each function:

(a) $f(x) = 3$
(b) $f(x) = x^2$
(c) $f(x) = \cos x$

Solution:

(a) $\int 3\,dx$

We need a function whose derivative is 3. Since $\frac{d}{dx}(3x) = 3$:

$$\int 3\,dx = 3x + C$$

Verification: $\frac{d}{dx}(3x + C) = 3$ ✓

(b) $\int x^2\,dx$

We need a function whose derivative is x^2. Since $\frac{d}{dx}\left(\frac{x^3}{3}\right) = x^2$:

$$\int x^2\,dx = \frac{x^3}{3} + C$$

Verification: $\frac{d}{dx}\left(\frac{x^3}{3} + C\right) = x^2$ ✓

(c) $\int \cos x\,dx$

We need a function whose derivative is $\cos x$. Since $\frac{d}{dx}(\sin x) = \cos x$:

$$\int \cos x\,dx = \sin x + C$$

Verification: $\frac{d}{dx}(\sin x + C) = \cos x$ ✓

Answer:

$$(a) \quad \boxed{\int 3\,dx = 3x + C}$$

$$(b) \quad \boxed{\int x^2\,dx = \frac{x^3}{3} + C}$$

$$(c) \quad \boxed{\int \cos x\,dx = \sin x + C}$$

Example 2: Antiderivatives with Initial Conditions

Problem: Find the function f such that $f'(x) = 2x + 1$ and $f(0) = 3$.

Solution:

Step 1: Find the general antiderivative

$$\begin{aligned} f(x) &= \int (2x+1)\,dx \\ &= \int 2x\,dx + \int 1\,dx \\ &= x^2 + x + C \end{aligned}$$

Step 2: Use the initial condition to find C

Given $f(0) = 3$:

$$\begin{aligned} f(0) &= 0^2 + 0 + C = 3 \\ C &= 3 \end{aligned}$$

Step 3: Write the specific solution

$$f(x) = x^2 + x + 3$$

Verification:

- $f'(x) = 2x + 1$ ✓
- $f(0) = 0 + 0 + 3 = 3$ ✓

Answer: $\boxed{f(x) = x^2 + x + 3}$

Key Idea: Initial conditions allow us to determine the specific value of C, giving us a **particular solution** rather than the general solution.

Definition 5.3: Notation and Conventions

Multiple Notations:

All of the following mean the same thing:

$$\int f(x)\,dx = F(x) + C$$
$$\int f = F + C$$
$$\text{Antiderivative of } f = F + C$$

Common Terminology:

- "Integrate f" means find $\int f(x)\,dx$
- "Find the antiderivative" means find $\int f(x)\,dx$
- "Evaluate the integral" means compute $\int f(x)\,dx$

Warning About Notation:

The differential dx is **essential**! It tells us:

- Which variable we're integrating with respect to
- Where the integrand ends

Examples:

$$\int x^2\,dx \neq \int x^2\,dt$$
$$\int 2x\,dx = x^2 + C$$
$$\int 2\,x\,dx = x^2 + C \quad \text{(same as above)}$$

Section 5.1 Summary

Key Definitions:

- **Antiderivative:** $F'(x) = f(x)$
- **Indefinite Integral:** $\int f(x)\,dx = F(x) + C$

Fundamental Relationship:

$$\boxed{\frac{d}{dx}\left[\int f(x)\,dx\right] = f(x)}$$

$$\boxed{\int f'(x)\,dx = f(x) + C}$$

Key Facts:

- Integration is the reverse of differentiation
- Antiderivatives differ by a constant
- Always include $+C$ for indefinite integrals

- Initial conditions determine specific value of C

Common Mistakes:

- Forgetting to add $+C$
- Forgetting the differential dx
- Confusing $\int f(x)\,dx$ with $\int f(x)\,dt$

Check Your Understanding:

- Can you explain why antiderivatives are not unique?
- What is the difference between $F(x)$ and $F(x)+C$?
- How do you verify an antiderivative is correct?

Basic Integration Rules

Just as we had rules for differentiation, we have corresponding rules for integration. These basic integration formulas are derived from differentiation rules and form the foundation for all integration techniques.

Theorem 5.2: Basic Integration Formulas

Power Rule for Integration:

$$\boxed{\int x^n\,dx = \frac{x^{n+1}}{n+1} + C \quad (n \neq -1)}$$

Special Cases:

$$\int k\,dx = kx + C \quad \text{(constant)}$$
$$\int \frac{1}{x}\,dx = \ln|x| + C \quad (n = -1 \text{ case})$$
$$\int e^x\,dx = e^x + C$$
$$\int a^x\,dx = \frac{a^x}{\ln a} + C \quad (a > 0, a \neq 1)$$

Trigonometric Integrals:

$$\int \sin x\,dx = -\cos x + C$$
$$\int \cos x\,dx = \sin x + C$$
$$\int \sec^2 x\,dx = \tan x + C$$
$$\int \csc^2 x\,dx = -\cot x + C$$
$$\int \sec x \tan x\,dx = \sec x + C$$
$$\int \csc x \cot x\,dx = -\csc x + C$$

How to Remember:

These formulas come directly from differentiation! For each derivative formula, there's a corresponding integral formula.

Watch the Signs:

Notice the negative signs in $\int \sin x\,dx = -\cos x + C$ and others. These come from the chain rule in reverse!

Theorem 5.3: Linearity Properties of Integration

Constant Multiple Rule:

$$\boxed{\int cf(x)\,dx = c\int f(x)\,dx}$$

Constants can be pulled out of integrals.

Sum Rule:

$$\boxed{\int [f(x) + g(x)]\, dx = \int f(x)\, dx + \int g(x)\, dx}$$

The integral of a sum is the sum of the integrals.

Difference Rule:

$$\boxed{\int [f(x) - g(x)]\, dx = \int f(x)\, dx - \int g(x)\, dx}$$

Combined:

For constants a and b:

$$\int [af(x) + bg(x)]\, dx = a\int f(x)\, dx + b\int g(x)\, dx$$

Warning:

There is **NO product rule or quotient rule for integration**!

$$\int f(x)g(x)\, dx \neq \left(\int f(x)\, dx\right)\left(\int g(x)\, dx\right)$$

Example 3: Using the Power Rule

Problem: Evaluate each integral:

(a) $\displaystyle\int x^5\, dx$

(b) $\displaystyle\int \frac{1}{x^3}\, dx$

(c) $\displaystyle\int \sqrt{x}\, dx$

Solution:

(a) $\displaystyle\int x^5\, dx$

Using the power rule with $n = 5$:

$$\int x^5\, dx = \frac{x^{5+1}}{5+1} + C = \frac{x^6}{6} + C$$

Verification: $\displaystyle\frac{d}{dx}\left(\frac{x^6}{6} + C\right) = x^5$ ✓

(b) $\displaystyle\int \frac{1}{x^3}\, dx = \int x^{-3}\, dx$

Using the power rule with $n = -3$:

$$\begin{aligned}\int x^{-3}\, dx &= \frac{x^{-3+1}}{-3+1} + C \\ &= \frac{x^{-2}}{-2} + C \\ &= -\frac{1}{2x^2} + C\end{aligned}$$

Verification: $\dfrac{d}{dx}\left(-\dfrac{1}{2x^2}+C\right) = -\dfrac{1}{2}\cdot(-2)x^{-3} = x^{-3} = \dfrac{1}{x^3}$ ✓

(c) $\displaystyle\int \sqrt{x}\,dx = \int x^{1/2}\,dx$

Using the power rule with $n = \dfrac{1}{2}$:

$$\begin{aligned}\int x^{1/2}\,dx &= \frac{x^{1/2+1}}{1/2+1}+C\\ &= \frac{x^{3/2}}{3/2}+C\\ &= \frac{2}{3}x^{3/2}+C\\ &= \frac{2}{3}x\sqrt{x}+C\end{aligned}$$

Answer:

$$(a)\quad \boxed{\frac{x^6}{6}+C}$$
$$(b)\quad \boxed{-\frac{1}{2x^2}+C}$$
$$(c)\quad \boxed{\frac{2}{3}x^{3/2}+C}$$

Example 4: Using Linearity Properties

Problem: Evaluate: $\displaystyle\int (3x^2-4x+5)\,dx$

Solution:

Step 1: Split using sum and difference rules

$$\int (3x^2-4x+5)\,dx = \int 3x^2\,dx - \int 4x\,dx + \int 5\,dx$$

Step 2: Pull out constants

$$= 3\int x^2\,dx - 4\int x\,dx + 5\int 1\,dx$$

Step 3: Apply power rule to each term

$$\begin{aligned}&= 3\cdot\frac{x^3}{3} - 4\cdot\frac{x^2}{2} + 5x + C\\ &= x^3 - 2x^2 + 5x + C\end{aligned}$$

Verification:

$$\frac{d}{dx}(x^3-2x^2+5x+C) = 3x^2-4x+5 \quad ✓$$

Answer: $\boxed{x^3 - 2x^2 + 5x + C}$

Note: Even though we had three integrals, we only need **one** constant C at the end!

Example 5: Rewriting Before Integrating

Problem: Evaluate each integral:

(a) $\int \frac{x^3 + 2x^2 - 5}{x^2}\, dx$

(b) $\int (2\sqrt{x} + \frac{3}{\sqrt{x}})\, dx$

Solution:

(a) $\int \frac{x^3 + 2x^2 - 5}{x^2}\, dx$

Step 1: Simplify by dividing each term

$$\begin{aligned}\int \frac{x^3 + 2x^2 - 5}{x^2}\, dx &= \int \left(\frac{x^3}{x^2} + \frac{2x^2}{x^2} - \frac{5}{x^2}\right) dx \\ &= \int \left(x + 2 - 5x^{-2}\right) dx\end{aligned}$$

Step 2: Integrate term by term

$$\begin{aligned}&= \frac{x^2}{2} + 2x - 5 \cdot \frac{x^{-1}}{-1} + C \\ &= \frac{x^2}{2} + 2x + \frac{5}{x} + C\end{aligned}$$

Answer (a): $\boxed{\frac{x^2}{2} + 2x + \frac{5}{x} + C}$

(b) $\int (2\sqrt{x} + \frac{3}{\sqrt{x}})\, dx$

Step 1: Rewrite using exponents

$$\int (2\sqrt{x} + \frac{3}{\sqrt{x}})\, dx = \int (2x^{1/2} + 3x^{-1/2})\, dx$$

Step 2: Integrate using power rule

$$\begin{aligned}&= 2 \cdot \frac{x^{3/2}}{3/2} + 3 \cdot \frac{x^{1/2}}{1/2} + C \\ &= \frac{4}{3}x^{3/2} + 6x^{1/2} + C \\ &= \frac{4}{3}x\sqrt{x} + 6\sqrt{x} + C\end{aligned}$$

Answer (b): $\boxed{\frac{4}{3}x^{3/2} + 6x^{1/2} + C}$

Strategy: When integrals look complicated, try:

- Simplifying algebraically first
- Rewriting roots as fractional exponents
- Splitting fractions into separate terms

Example 6: Trigonometric Integrals

Problem: Evaluate:

(a) $\int (3\sin x - 2\cos x)\,dx$

(b) $\int (4\sec^2 x + \sec x \tan x)\,dx$

Solution:

(a) $\int (3\sin x - 2\cos x)\,dx$

$$\begin{aligned}\int (3\sin x - 2\cos x)\,dx &= 3\int \sin x\,dx - 2\int \cos x\,dx \\ &= 3(-\cos x) - 2(\sin x) + C \\ &= -3\cos x - 2\sin x + C\end{aligned}$$

Answer (a): $\boxed{-3\cos x - 2\sin x + C}$

(b) $\int (4\sec^2 x + \sec x \tan x)\,dx$

$$\begin{aligned}\int (4\sec^2 x + \sec x\tan x)\,dx &= 4\int \sec^2 x\,dx + \int \sec x \tan x\,dx \\ &= 4\tan x + \sec x + C\end{aligned}$$

Answer (b): $\boxed{4\tan x + \sec x + C}$

Verification for (a):

$$\frac{d}{dx}(-3\cos x - 2\sin x + C) = 3\sin x - 2\cos x \quad ✓$$

Example 7: Exponential and Logarithmic Integrals

Problem: Evaluate:

(a) $\int (e^x + 2^x)\,dx$

(b) $\int \frac{3}{x}\,dx$

Solution:

(a) $\int (e^x + 2^x)\,dx$

$$\begin{aligned}\int (e^x + 2^x)\,dx &= \int e^x\,dx + \int 2^x\,dx \\ &= e^x + \frac{2^x}{\ln 2} + C\end{aligned}$$

Answer (a): $\boxed{e^x + \dfrac{2^x}{\ln 2} + C}$

(b) $\int \dfrac{3}{x}\,dx$

$$\begin{aligned}\int \frac{3}{x}\,dx &= 3\int \frac{1}{x}\,dx \\ &= 3\ln|x| + C\end{aligned}$$

Answer (b): $\boxed{3\ln|x| + C}$

⚠ **Important:** For $\int \frac{1}{x}\,dx$, always use $\ln|x|$ (with absolute value) to handle both positive and negative x.

Section 5.2 Summary: Basic Integration Rules

Essential Formulas:

Function	Integral
x^n $(n \neq -1)$	$\dfrac{x^{n+1}}{n+1} + C$
$\dfrac{1}{x}$	$\ln|x| + C$
e^x	$e^x + C$
a^x	$\dfrac{a^x}{\ln a} + C$
$\sin x$	$-\cos x + C$
$\cos x$	$\sin x + C$
$\sec^2 x$	$\tan x + C$
$\sec x \tan x$	$\sec x + C$

Linearity Rules:

$$\int cf(x)\,dx = c\int f(x)\,dx$$
$$\int [f(x) \pm g(x)]\,dx = \int f(x)\,dx \pm \int g(x)\,dx$$

Integration Strategy:

1. Simplify the integrand algebraically
2. Rewrite roots as fractional exponents
3. Split into sum of simpler terms
4. Apply appropriate integration formulas
5. Don't forget $+C$!

⚠ **Common Mistakes:**

- Forgetting $+C$
- Using $\ln x$ instead of $\ln|x|$ for $\int \frac{1}{x}dx$

- Trying to apply product/quotient rules (they don't exist!)
- Wrong sign in $\int \sin x \, dx = -\cos x + C$ (not $+\cos x$)
- Forgetting to simplify before integrating

Always Verify: Differentiate your answer to check if you get back the original integrand!

Integration by Substitution

Integration by substitution (also called u-substitution) is the integration counterpart of the chain rule for differentiation. It is one of the most important and frequently used integration techniques.

Theorem 5.4: Substitution Rule (Change of Variable)

If $u = g(x)$ is a differentiable function whose range is an interval I, and f is continuous on I, then:

$$\boxed{\int f(g(x))g'(x)\,dx = \int f(u)\,du}$$

where $u = g(x)$ and $du = g'(x)\,dx$.

Equivalent Form:

After finding the antiderivative in terms of u, substitute back:

$$\int f(g(x))g'(x)\,dx = F(g(x)) + C$$

where F is an antiderivative of f.

Why It Works:

By the chain rule:

$$\frac{d}{dx}[F(g(x))] = F'(g(x)) \cdot g'(x) = f(g(x)) \cdot g'(x)$$

So integration by substitution is just the chain rule "backwards"!

Key Recognition:

Look for integrals of the form:

$$\int [\text{function}] \times [\text{derivative of inside}]\,dx$$

Definition 5.4: Step-by-Step Substitution Method

Step 1: Choose u

- Let u equal the "inside function" (usually the more complicated part)
- Look for a function whose derivative appears elsewhere in the integrand

Step 2: Find du

- Differentiate: $du = g'(x)\,dx$
- If necessary, solve for dx: $dx = \frac{du}{g'(x)}$

Step 3: Substitute

- Replace all x terms with u terms
- Replace dx with du (or appropriate multiple)
- Check: the integral should now be entirely in terms of u

Step 4: Integrate

- Integrate with respect to u
- Don't forget $+C$

Step 5: Back-substitute

- Replace u with the original expression in x
- Final answer should be in terms of x only

When to Use Substitution:

- Composition of functions (one inside another)
- You see a function and its derivative (or a constant multiple)
- Complex expressions that would simplify with substitution

Example 8: Basic Substitution

Problem: Evaluate $\int 2x(x^2+1)^5\,dx$

Solution:
Step 1: Choose u
Let $u = x^2 + 1$ (the inside function)
Step 2: Find du

$$\frac{du}{dx} = 2x$$
$$du = 2x\,dx$$

Step 3: Substitute
Notice that $2x\,dx$ appears in the original integral!

$$\int 2x(x^2+1)^5\,dx = \int (x^2+1)^5 \cdot 2x\,dx$$
$$= \int u^5\,du$$

Step 4: Integrate

$$\int u^5\,du = \frac{u^6}{6} + C$$

Step 5: Back-substitute

$$= \frac{(x^2+1)^6}{6} + C$$

Answer: $\boxed{\frac{(x^2+1)^6}{6} + C}$
Verification:

$$\frac{d}{dx}\left[\frac{(x^2+1)^6}{6}\right] = \frac{6(x^2+1)^5 \cdot 2x}{6} = 2x(x^2+1)^5 \quad \checkmark$$

Example 9: Substitution with Adjustment

Problem: Evaluate $\int x(x^2+3)^4\,dx$

Solution:
Step 1: Choose u
Let $u = x^2 + 3$
Step 2: Find du

$$du = 2x\,dx$$
$$x\,dx = \frac{1}{2}du$$

Step 3: Substitute

$$\begin{aligned}\int x(x^2+3)^4\,dx &= \int (x^2+3)^4 \cdot x\,dx \\ &= \int u^4 \cdot \frac{1}{2}du \\ &= \frac{1}{2}\int u^4\,du\end{aligned}$$

Step 4: Integrate

$$\begin{aligned}\frac{1}{2}\int u^4\,du &= \frac{1}{2}\cdot\frac{u^5}{5} + C \\ &= \frac{u^5}{10} + C\end{aligned}$$

Step 5: Back-substitute

$$= \frac{(x^2+3)^5}{10} + C$$

Answer: $\boxed{\frac{(x^2+3)^5}{10} + C}$

Key Point: Even though we had $x\,dx$ but needed $2x\,dx$, we adjusted by multiplying by $\frac{1}{2}$.

Example 10: Trigonometric Substitution

Problem: Evaluate $\int \sin^3 x\cos x\,dx$

Solution:
Step 1: Choose u
Let $u = \sin x$ (note: $\cos x$ is the derivative of $\sin x$)
Step 2: Find du

$$du = \cos x\, dx$$

Step 3: Substitute

$$\begin{aligned}\int \sin^3 x \cos x\, dx &= \int (\sin x)^3 \cdot \cos x\, dx \\ &= \int u^3\, du\end{aligned}$$

Step 4: Integrate

$$\int u^3\, du = \frac{u^4}{4} + C$$

Step 5: Back-substitute

$$= \frac{\sin^4 x}{4} + C$$

Answer: $\boxed{\frac{\sin^4 x}{4} + C}$

Verification:

$$\frac{d}{dx}\left[\frac{\sin^4 x}{4}\right] = \frac{4\sin^3 x \cos x}{4} = \sin^3 x \cos x \quad \checkmark$$

Example 11: Exponential with Substitution

Problem: Evaluate $\int xe^{x^2}\, dx$

Solution:

Step 1: Choose u

Let $u = x^2$ (the exponent)

Step 2: Find du

$$\begin{aligned}du &= 2x\, dx \\ x\, dx &= \frac{1}{2} du\end{aligned}$$

Step 3: Substitute

$$\begin{aligned}\int xe^{x^2}\, dx &= \int e^{x^2} \cdot x\, dx \\ &= \int e^u \cdot \frac{1}{2} du \\ &= \frac{1}{2}\int e^u\, du\end{aligned}$$

Step 4: Integrate

$$\frac{1}{2}\int e^u\,du = \frac{1}{2}e^u + C$$

Step 5: Back-substitute

$$= \frac{1}{2}e^{x^2} + C$$

Answer: $\boxed{\frac{1}{2}e^{x^2} + C}$

Example 12: Substitution with Logarithms

Problem: Evaluate $\int \frac{2x+3}{x^2+3x+5}\,dx$

Solution:
Step 1: Recognize the pattern
Notice: derivative of denominator is $2x+3$ (exactly the numerator!)
Let $u = x^2 + 3x + 5$
Step 2: Find du

$$du = (2x+3)\,dx$$

Step 3: Substitute

$$\int \frac{2x+3}{x^2+3x+5}\,dx = \int \frac{1}{u}\,du$$

Step 4: Integrate

$$\int \frac{1}{u}\,du = \ln|u| + C$$

Step 5: Back-substitute

$$= \ln|x^2 + 3x + 5| + C$$

Since $x^2 + 3x + 5 = (x + \frac{3}{2})^2 + \frac{11}{4} > 0$ always, we can drop absolute value:
Answer: $\boxed{\ln(x^2 + 3x + 5) + C}$

Pattern: Whenever you see $\int \frac{f'(x)}{f(x)}\,dx$, the answer is $\ln|f(x)| + C$

Example 13: More Complex Substitution

Problem: Evaluate $\int \frac{x}{\sqrt{x^2+1}}\,dx$

Solution:

Step 1: Choose u

Let $u = x^2 + 1$

Step 2: Find du

$$\begin{aligned} du &= 2x\,dx \\ x\,dx &= \frac{1}{2}du \end{aligned}$$

Step 3: Substitute

$$\begin{aligned} \int \frac{x}{\sqrt{x^2+1}}\,dx &= \int \frac{1}{\sqrt{x^2+1}} \cdot x\,dx \\ &= \int \frac{1}{\sqrt{u}} \cdot \frac{1}{2}du \\ &= \frac{1}{2}\int u^{-1/2}\,du \end{aligned}$$

Step 4: Integrate

$$\begin{aligned} \frac{1}{2}\int u^{-1/2}\,du &= \frac{1}{2} \cdot \frac{u^{1/2}}{1/2} + C \\ &= u^{1/2} + C \\ &= \sqrt{u} + C \end{aligned}$$

Step 5: Back-substitute

$$= \sqrt{x^2+1} + C$$

Answer: $\boxed{\sqrt{x^2+1}+C}$

Verification:

$$\frac{d}{dx}[\sqrt{x^2+1}] = \frac{2x}{2\sqrt{x^2+1}} = \frac{x}{\sqrt{x^2+1}} \quad \checkmark$$

Section 5.3 Summary: Integration by Substitution

Substitution Rule:

$$\boxed{\int f(g(x))g'(x)\,dx = \int f(u)\,du \quad \text{where } u = g(x)}$$

5-Step Method:

1. Choose u (usually the inside function)

2. Find $du = g'(x)\,dx$
3. Substitute (convert everything to u)
4. Integrate with respect to u
5. Back-substitute (replace u with expression in x)

Common Patterns:

- $\int f(g(x))g'(x)\,dx \to$ let $u = g(x)$
- $\int \frac{f'(x)}{f(x)}\,dx = \ln|f(x)| + C$
- Look for function and its derivative

Exam Alert: u-substitution appears in **almost every exam**! Master this technique.
⚠ **Common Mistakes:**

- Not adjusting coefficients (forgetting the $\frac{1}{2}$, etc.)
- Forgetting to back-substitute at the end
- Mixing x and u in the same integral
- Not completely converting to u before integrating

Practice Tip: Always verify by differentiating your answer!

Integration by Parts

Integration by parts is the integration counterpart of the product rule for differentiation. It's used when the integrand is a product of two functions, and substitution doesn't work.

Theorem 5.5: Integration by Parts Formula

If u and v are differentiable functions of x, then:

$$\boxed{\int u\,dv = uv - \int v\,du}$$

Derivation from Product Rule:

Starting with the product rule:

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$$

Rearranging:

$$u\frac{dv}{dx} = \frac{d}{dx}(uv) - v\frac{du}{dx}$$

Integrating both sides:

$$\int u\frac{dv}{dx}\,dx = \int \frac{d}{dx}(uv)\,dx - \int v\frac{du}{dx}\,dx$$
$$\int u\,dv = uv - \int v\,du$$

Alternative Notation:

$$\int f(x)g(x)\,dx = f(x)G(x) - \int f'(x)G(x)\,dx$$

where $G(x)$ is an antiderivative of $g(x)$.

Definition 5.5: Step-by-Step Integration by Parts

Step 1: Identify u and dv

- Choose which part to call u and which to call dv
- Use the LIATE rule (see below) as a guide
- Must have: $\int u\,dv = \int [\text{product}]\,dx$

Step 2: Compute du and v

- Differentiate: $du = u'\,dx$
- Integrate: $v = \int dv$

Step 3: Apply the formula

$$\int u\,dv = uv - \int v\,du$$

Step 4: Evaluate $\int v\,du$

- This should be simpler than the original integral
- If not, try different choices for u and dv
- Sometimes need to apply parts multiple times

Step 5: Simplify and add $+C$

LIATE Rule for Choosing u:
When the integrand is a product, choose u in this priority order:

L	Logarithmic functions ($\ln x$, $\log x$)
I	Inverse trig functions ($\arcsin x$, $\arctan x$)
A	Algebraic functions (x, x^2, $\sqrt{x}$)
T	Trigonometric functions ($\sin x$, $\cos x$)
E	Exponential functions (e^x, 2^x)

Why LIATE?
Functions higher in the list simplify when differentiated; those lower simplify (or stay the same) when integrated.

Example 14: Basic Integration by Parts

Problem: Evaluate $\int xe^x\,dx$

Solution:
Step 1: Choose u and dv
Using LIATE: x is algebraic (A), e^x is exponential (E)
Choose: $u = x$ and $dv = e^x\,dx$
Step 2: Compute du and v

$$\begin{aligned} du &= dx \\ v &= \int e^x\,dx = e^x \end{aligned}$$

Step 3: Apply formula

$$\begin{aligned} \int xe^x\,dx &= uv - \int v\,du \\ &= xe^x - \int e^x\,dx \end{aligned}$$

Step 4: Evaluate remaining integral

$$= xe^x - e^x + C$$

Step 5: Simplify

$$= e^x(x-1) + C$$

Answer: $\boxed{e^x(x-1)+C}$
Verification:

$$\frac{d}{dx}[e^x(x-1)] = e^x(x-1) + e^x \cdot 1 = e^x \cdot x \quad \checkmark$$

Example 15: Integration by Parts with Polynomial

Problem: Evaluate $\displaystyle\int x \sin x \, dx$

Solution:
Step 1: Choose u and dv
Using LIATE: x is algebraic (A), $\sin x$ is trig (T)
Choose: $u = x$ and $dv = \sin x \, dx$
Step 2: Compute du and v

$$\begin{aligned} du &= dx \\ v &= \int \sin x \, dx = -\cos x \end{aligned}$$

Step 3: Apply formula

$$\begin{aligned} \int x \sin x \, dx &= uv - \int v \, du \\ &= x(-\cos x) - \int (-\cos x) \, dx \\ &= -x \cos x + \int \cos x \, dx \end{aligned}$$

Step 4: Evaluate

$$= -x \cos x + \sin x + C$$

Answer: $\boxed{-x\cos x + \sin x + C}$
Verification:

$$\frac{d}{dx}[-x\cos x + \sin x] = -\cos x + x \sin x + \cos x = x \sin x \quad \checkmark$$

Example 16: Integration by Parts with Logarithm

Problem: Evaluate $\displaystyle\int \ln x \, dx$

Solution:
Step 1: Rewrite and choose
Rewrite as: $\int \ln x \cdot 1 \, dx$
Using LIATE: $\ln x$ is logarithmic (L), 1 is algebraic (A)
Choose: $u = \ln x$ and $dv = 1 \, dx$

Step 2: Compute du and v

$$\begin{aligned} du &= \frac{1}{x}\,dx \\ v &= \int 1\,dx = x \end{aligned}$$

Step 3: Apply formula

$$\begin{aligned} \int \ln x\,dx &= uv - \int v\,du \\ &= x\ln x - \int x \cdot \frac{1}{x}\,dx \\ &= x\ln x - \int 1\,dx \end{aligned}$$

Step 4: Evaluate

$$= x\ln x - x + C$$

Answer: $\boxed{x\ln x - x + C}$

Key Technique: For $\int \ln x\,dx$, treat $\ln x$ as the "u" and let $dv = dx$. This is a standard trick for logarithms!

Example 17: Repeated Integration by Parts

Problem: Evaluate $\displaystyle\int x^2 e^x\,dx$

Solution:

First Application:

Choose: $u = x^2$, $dv = e^x\,dx$

$$\begin{aligned} du &= 2x\,dx \\ v &= e^x \end{aligned}$$

Apply formula:

$$\begin{aligned} \int x^2 e^x\,dx &= x^2 e^x - \int e^x \cdot 2x\,dx \\ &= x^2 e^x - 2\int x e^x\,dx \end{aligned}$$

Second Application:

For $\int x e^x\,dx$, choose: $u = x$, $dv = e^x\,dx$

$$\begin{aligned} du &= dx \\ v &= e^x \end{aligned}$$

Apply formula:

$$\int xe^x\,dx = xe^x - \int e^x\,dx$$
$$= xe^x - e^x$$

Combine:

$$\int x^2e^x\,dx = x^2e^x - 2(xe^x - e^x) + C$$
$$= x^2e^x - 2xe^x + 2e^x + C$$
$$= e^x(x^2 - 2x + 2) + C$$

Answer: $\boxed{e^x(x^2 - 2x + 2) + C}$

Pattern: When integrating x^ne^x, you need to apply parts n times!

Example 18: Tabular Integration (Shortcut)

Problem: Evaluate $\int x^3 \sin x\,dx$ using tabular method.

Solution:

Tabular Method: (useful when one function eventually becomes zero upon differentiation)

Sign	**Differentiate (u)**	**Integrate (dv)**
$+$	x^3	$\sin x$
$-$	$3x^2$	$-\cos x$
$+$	$6x$	$-\sin x$
$-$	6	$\cos x$
$+$	0	$\sin x$

Multiply diagonally with alternating signs:

$$\int x^3 \sin x\,dx = (+)(x^3)(-\cos x) + (-)(3x^2)(-\sin x)$$
$$+ (+)(6x)(\cos x) + (-)(6)(\sin x) + C$$
$$= -x^3\cos x + 3x^2\sin x + 6x\cos x - 6\sin x + C$$

Answer: $\boxed{-x^3\cos x + 3x^2\sin x + 6x\cos x - 6\sin x + C}$

When to Use Tabular Method:

- Polynomial times sine, cosine, or exponential
- One function becomes zero after repeated differentiation
- Saves time compared to repeated integration by parts

Example 19: Cyclic Integration by Parts

Problem: Evaluate $\int e^x \sin x \, dx$

Solution:

First Application:

Choose: $u = \sin x$, $dv = e^x \, dx$

$$\begin{aligned} du &= \cos x \, dx \\ v &= e^x \end{aligned}$$

$$\int e^x \sin x \, dx = e^x \sin x - \int e^x \cos x \, dx \quad (*)$$

Second Application:

For $\int e^x \cos x \, dx$, choose: $u = \cos x$, $dv = e^x \, dx$

$$\begin{aligned} du &= -\sin x \, dx \\ v &= e^x \end{aligned}$$

$$\begin{aligned} \int e^x \cos x \, dx &= e^x \cos x - \int e^x (-\sin x) \, dx \\ &= e^x \cos x + \int e^x \sin x \, dx \end{aligned}$$

Substitute back into (*):

$$\begin{aligned} \int e^x \sin x \, dx &= e^x \sin x - \left(e^x \cos x + \int e^x \sin x \, dx \right) \\ \int e^x \sin x \, dx &= e^x \sin x - e^x \cos x - \int e^x \sin x \, dx \end{aligned}$$

Solve for the integral:

$$\begin{aligned} 2 \int e^x \sin x \, dx &= e^x \sin x - e^x \cos x \\ \int e^x \sin x \, dx &= \frac{e^x (\sin x - \cos x)}{2} + C \end{aligned}$$

Answer: $\boxed{\frac{e^x (\sin x - \cos x)}{2} + C}$

Cyclic Technique: When the integral reappears, bring it to one side and solve algebraically!

Section 5.4 Summary: Integration by Parts

Integration by Parts Formula:

$$\boxed{\int u\,dv = uv - \int v\,du}$$

LIATE Rule (for choosing u):

Logarithmic > **I**nverse trig > **A**lgebraic > **T**rig > **E**xponential

When to Use Integration by Parts:

- Product of two functions
- Substitution doesn't work
- Functions from different categories (poly × trig, poly × exp, etc.)
- Logarithmic or inverse trig functions (even alone!)

Special Cases:

- $\int \ln x\,dx = x\ln x - x + C$
- $\int x^n e^x\,dx$ requires n applications
- $\int e^x \sin x\,dx$ or $\int e^x \cos x\,dx$ use cyclic technique

Common Mistakes:

- Wrong choice of u and dv (violating LIATE)
- Forgetting to include $-\int v\,du$ term
- Sign errors (especially with trig functions)
- Not recognizing when integral is getting more complicated (try different u)
- In cyclic method: forgetting to bring integral to one side

Pro Tips:

- Always verify by differentiation
- Tabular method saves time for repeated parts
- If integral gets worse, try different u and dv
- Watch for cyclic patterns with $e^x \sin x$ or $e^x \cos x$

Trigonometric Integrals

Trigonometric integrals involve products and powers of trigonometric functions. Success with these integrals depends on recognizing patterns and applying appropriate trigonometric identities.

Theorem 5.6: Key Trigonometric Identities for Integration

Pythagorean Identities:

$$\sin^2 x + \cos^2 x = 1$$
$$1 + \tan^2 x = \sec^2 x$$
$$1 + \cot^2 x = \csc^2 x$$

Double Angle Formulas:

$$\sin^2 x = \frac{1 - \cos(2x)}{2}$$
$$\cos^2 x = \frac{1 + \cos(2x)}{2}$$
$$\sin x \cos x = \frac{\sin(2x)}{2}$$

Product-to-Sum Formulas:

$$\sin A \sin B = \frac{1}{2}[\cos(A - B) - \cos(A + B)]$$
$$\cos A \cos B = \frac{1}{2}[\cos(A - B) + \cos(A + B)]$$
$$\sin A \cos B = \frac{1}{2}[\sin(A + B) + \sin(A - B)]$$

When to Use:

- Use Pythagorean identities to convert between sin and cos
- Use double angle formulas to reduce powers
- Use product-to-sum to eliminate products

Definition 5.6: Strategies for Trigonometric Integrals

Type 1: $\int \sin^m x \cos^n x \, dx$

Case 1a: If m is odd

- Save one $\sin x$ for du
- Convert rest using $\sin^2 x = 1 - \cos^2 x$
- Substitute $u = \cos x$

Case 1b: If n is odd

- Save one $\cos x$ for du
- Convert rest using $\cos^2 x = 1 - \sin^2 x$

- Substitute $u = \sin x$

Case 1c: If both m and n are even

- Use power-reducing formulas:

$$\sin^2 x = \frac{1 - \cos(2x)}{2}$$
$$\cos^2 x = \frac{1 + \cos(2x)}{2}$$

Type 2: $\int \tan^m x \sec^n x \, dx$
Case 2a: If n is even

- Save $\sec^2 x$ for du
- Convert rest using $\sec^2 x = 1 + \tan^2 x$
- Substitute $u = \tan x$

Case 2b: If m is odd

- Save $\sec x \tan x$ for du
- Convert rest using $\tan^2 x = \sec^2 x - 1$
- Substitute $u = \sec x$

Case 2c: Special cases

- $\int \tan x \, dx = -\ln|\cos x| + C = \ln|\sec x| + C$
- $\int \sec x \, dx = \ln|\sec x + \tan x| + C$

Example 20: $\sin^m x \cos^n x$ with Odd Power

Problem: Evaluate $\displaystyle\int \sin^3 x \cos^2 x \, dx$

Solution:
Strategy: Power of $\sin x$ is odd (3), so save one $\sin x$ for du.
Step 1: Rewrite

$$\int \sin^3 x \cos^2 x \, dx = \int \sin^2 x \cos^2 x \cdot \sin x \, dx$$

Step 2: Convert $\sin^2 x$ using Pythagorean identity

$$= \int (1 - \cos^2 x) \cos^2 x \cdot \sin x \, dx$$

Step 3: Substitute $u = \cos x$, $du = -\sin x \, dx$

$$= \int (1-u^2)u^2 \cdot (-du)$$
$$= -\int (u^2 - u^4)\, du$$
$$= -\int u^2\, du + \int u^4\, du$$

Step 4: Integrate

$$= -\frac{u^3}{3} + \frac{u^5}{5} + C$$

Step 5: Back-substitute

$$= -\frac{\cos^3 x}{3} + \frac{\cos^5 x}{5} + C$$

Answer: $\boxed{\frac{\cos^5 x}{5} - \frac{\cos^3 x}{3} + C}$

Example 21: $\sin^m x \cos^n x$ with Both Even

Problem: Evaluate $\int \sin^2 x \cos^2 x\, dx$

Solution:
Strategy: Both powers are even, use power-reducing formulas.
Step 1: Apply double angle formulas

$$\sin^2 x = \frac{1-\cos(2x)}{2}$$
$$\cos^2 x = \frac{1+\cos(2x)}{2}$$

Step 2: Multiply

$$\int \sin^2 x \cos^2 x\, dx = \int \frac{1-\cos(2x)}{2} \cdot \frac{1+\cos(2x)}{2}\, dx$$
$$= \frac{1}{4}\int (1-\cos(2x))(1+\cos(2x))\, dx$$
$$= \frac{1}{4}\int (1-\cos^2(2x))\, dx$$

Step 3: Use identity again: $\cos^2(2x) = \dfrac{1+\cos(4x)}{2}$

$$= \frac{1}{4}\int \left(1 - \frac{1+\cos(4x)}{2}\right) dx$$
$$= \frac{1}{4}\int \left(\frac{1-\cos(4x)}{2}\right) dx$$
$$= \frac{1}{8}\int (1-\cos(4x))\, dx$$

Step 4: Integrate

$$= \frac{1}{8}\left(x - \frac{\sin(4x)}{4}\right) + C$$
$$= \frac{x}{8} - \frac{\sin(4x)}{32} + C$$

Answer: $\boxed{\frac{x}{8} - \frac{\sin(4x)}{32} + C}$

Alternative: Could also use $\sin x \cos x = \frac{\sin(2x)}{2}$, then $\sin^2(2x) = \frac{1-\cos(4x)}{2}$.

Example 22: $\tan^m x \sec^n x$ with Even Secant

Problem: Evaluate $\int \tan^2 x \sec^4 x\, dx$

Solution:
Strategy: Power of $\sec x$ is even (4), save $\sec^2 x$ for du.
Step 1: Rewrite

$$\int \tan^2 x \sec^4 x\, dx = \int \tan^2 x \sec^2 x \cdot \sec^2 x\, dx$$

Step 2: Convert $\sec^2 x$ using identity $\sec^2 x = 1 + \tan^2 x$

$$= \int \tan^2 x(1+\tan^2 x)\cdot \sec^2 x\, dx$$

Step 3: Substitute $u = \tan x$, $du = \sec^2 x\, dx$

$$= \int u^2(1+u^2)\, du$$
$$= \int (u^2+u^4)\, du$$

Step 4: Integrate

$$= \frac{u^3}{3} + \frac{u^5}{5} + C$$

Step 5: Back-substitute

$$= \frac{\tan^3 x}{3} + \frac{\tan^5 x}{5} + C$$

Answer: $\boxed{\frac{\tan^3 x}{3} + \frac{\tan^5 x}{5} + C}$

Example 23: Special Integral – $\int \sec x \, dx$

Problem: Evaluate $\displaystyle\int \sec x \, dx$

Solution:
Strategy: Multiply by $\dfrac{\sec x + \tan x}{\sec x + \tan x}$ (clever trick!)
Step 1: Multiply by conjugate

$$\int \sec x \, dx = \int \sec x \cdot \frac{\sec x + \tan x}{\sec x + \tan x} \, dx$$
$$= \int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x} \, dx$$

Step 2: Substitute $u = \sec x + \tan x$
Then: $du = (\sec x \tan x + \sec^2 x) \, dx$ (exactly the numerator!)

$$= \int \frac{1}{u} \, du$$

Step 3: Integrate

$$= \ln |u| + C$$

Step 4: Back-substitute

$$= \ln |\sec x + \tan x| + C$$

Answer: $\boxed{\ln |\sec x + \tan x| + C}$
Memorize: This is a standard result worth remembering!

Example 24: Product of Different Trig Functions

Problem: Evaluate $\displaystyle\int \sin(3x) \cos(5x) \, dx$

Solution:
Strategy: Use product-to-sum formula
Step 1: Apply product-to-sum

$$\sin A \cos B = \frac{1}{2}[\sin(A+B) + \sin(A-B)]$$

With $A = 3x$ and $B = 5x$:

$$\begin{aligned}\sin(3x)\cos(5x) &= \frac{1}{2}[\sin(8x) + \sin(-2x)] \\ &= \frac{1}{2}[\sin(8x) - \sin(2x)]\end{aligned}$$

Step 2: Integrate

$$\begin{aligned}\int \sin(3x)\cos(5x)\,dx &= \frac{1}{2}\int [\sin(8x) - \sin(2x)]\,dx \\ &= \frac{1}{2}\left[-\frac{\cos(8x)}{8} + \frac{\cos(2x)}{2}\right] + C \\ &= -\frac{\cos(8x)}{16} + \frac{\cos(2x)}{4} + C\end{aligned}$$

Answer: $\boxed{-\frac{\cos(8x)}{16} + \frac{\cos(2x)}{4} + C}$

Section 5.5 Summary: Trigonometric Integrals

Strategy Summary:

Integral Type	**Strategy**
$\int \sin^m x \cos^n x\,dx$ **Both even:** Power reduction	**Odd power:** Save one, convert rest
$\int \tan^m x \sec^n x\,dx$ **Odd tan:** Save $\sec x \tan x$, let $u = \sec x$	**Even sec:** Save $\sec^2 x$, let $u = \tan x$
Products of different args	Use product-to-sum formulas

Essential Formulas to Memorize:

$$\int \tan x\,dx = \ln|\sec x| + C$$

$$\int \sec x\,dx = \ln|\sec x + \tan x| + C$$

$$\int \csc x\,dx = -\ln|\csc x + \cot x| + C$$

Power-Reducing Formulas:

$$\sin^2 x = \frac{1-\cos(2x)}{2}$$

$$\cos^2 x = \frac{1+\cos(2x)}{2}$$

Common Mistakes:

- Forgetting to save a factor for du
- Not recognizing when to use power reduction
- Sign errors with $\sin^2 x = 1 - \cos^2 x$
- Forgetting chain rule when integrating (e.g., $\sin(2x)$)

Partial Fractions

Partial fraction decomposition is a technique for integrating rational functions (ratios of polynomials). The idea is to break a complex fraction into simpler pieces that can be integrated individually.

Definition 5.7: Partial Fraction Decomposition

A **rational function** has the form:

$$\frac{P(x)}{Q(x)}$$

where $P(x)$ and $Q(x)$ are polynomials.

Proper vs. Improper:

- **Proper:** $\deg(P) < \deg(Q)$ – can use partial fractions directly
- **Improper:** $\deg(P) \geq \deg(Q)$ – first perform polynomial long division

Basic Idea:

Break down a complicated fraction:

$$\frac{2x+1}{(x-1)(x+3)} = \frac{A}{x-1} + \frac{B}{x+3}$$

into simpler fractions that are easy to integrate.

Theorem 5.7: Cases for Partial Fraction Decomposition

Case 1: Distinct Linear Factors

If $Q(x) = (x-a_1)(x-a_2)\cdots(x-a_n)$ with all a_i distinct:

$$\frac{P(x)}{Q(x)} = \frac{A_1}{x-a_1} + \frac{A_2}{x-a_2} + \cdots + \frac{A_n}{x-a_n}$$

Case 2: Repeated Linear Factors

If $Q(x)$ contains $(x-a)^k$:

$$\frac{P(x)}{Q(x)} = \frac{A_1}{x-a} + \frac{A_2}{(x-a)^2} + \cdots + \frac{A_k}{(x-a)^k} + \text{(other terms)}$$

Case 3: Distinct Irreducible Quadratic Factors

If $Q(x)$ contains ax^2+bx+c (which cannot be factored):

$$\frac{P(x)}{Q(x)} = \frac{Ax+B}{ax^2+bx+c} + \text{(other terms)}$$

Case 4: Repeated Irreducible Quadratic Factors

If $Q(x)$ contains $(ax^2+bx+c)^k$:

$$\frac{P(x)}{Q(x)} = \frac{A_1x+B_1}{ax^2+bx+c} + \frac{A_2x+B_2}{(ax^2+bx+c)^2} + \cdots + \text{(other terms)}$$

Strategy:

1. Factor the denominator completely
2. Set up partial fraction form based on factors
3. Multiply both sides by denominator
4. Solve for unknown constants
5. Integrate each term separately

Example 25: Distinct Linear Factors

Problem: Evaluate $\int \frac{5x-2}{x^2-x-2}\,dx$

Solution:
Step 1: Factor denominator

$$x^2 - x - 2 = (x-2)(x+1)$$

Step 2: Set up partial fractions

$$\frac{5x-2}{(x-2)(x+1)} = \frac{A}{x-2} + \frac{B}{x+1}$$

Step 3: Multiply by $(x-2)(x+1)$

$$5x - 2 = A(x+1) + B(x-2)$$

Step 4: Solve for A and B
Method 1: Substitution
Let $x = 2$: $5(2) - 2 = A(3) + 0 \implies 8 = 3A \implies A = \frac{8}{3}$
Let $x = -1$: $5(-1) - 2 = 0 + B(-3) \implies -7 = -3B \implies B = \frac{7}{3}$
Method 2: Compare coefficients
$5x - 2 = Ax + A + Bx - 2B = (A+B)x + (A-2B)$
Coefficients of x: $5 = A + B$
Constants: $-2 = A - 2B$
Solving: $A = \frac{8}{3}$, $B = \frac{7}{3}$
Step 5: Integrate

$$\begin{aligned}\int \frac{5x-2}{x^2-x-2}\,dx &= \int \left(\frac{8/3}{x-2} + \frac{7/3}{x+1}\right) dx \\ &= \frac{8}{3}\int \frac{1}{x-2}\,dx + \frac{7}{3}\int \frac{1}{x+1}\,dx \\ &= \frac{8}{3}\ln|x-2| + \frac{7}{3}\ln|x+1| + C\end{aligned}$$

Answer: $\boxed{\frac{8}{3}\ln|x-2| + \frac{7}{3}\ln|x+1| + C}$

Example 26: Repeated Linear Factor

Problem: Evaluate $\int \frac{2x}{(x-1)^2(x+2)}\,dx$

Solution:

Step 1: Set up partial fractions (repeated factor $(x-1)^2$)

$$\frac{2x}{(x-1)^2(x+2)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x+2}$$

Step 2: Multiply by $(x-1)^2(x+2)$

$$2x = A(x-1)(x+2) + B(x+2) + C(x-1)^2$$

Step 3: Solve for constants

Let $x = 1$: $2 = 0 + B(3) + 0 \implies B = \frac{2}{3}$

Let $x = -2$: $-4 = 0 + 0 + C(9) \implies C = -\frac{4}{9}$

Let $x = 0$: $0 = A(-1)(2) + \frac{2}{3}(2) + (-\frac{4}{9})(1)$

$0 = -2A + \frac{4}{3} - \frac{4}{9}$

$0 = -2A + \frac{12-4}{9} = -2A + \frac{8}{9}$

$A = \frac{4}{9}$

Step 4: Integrate

$$\begin{aligned}\int \frac{2x}{(x-1)^2(x+2)}\,dx &= \int \left(\frac{4/9}{x-1} + \frac{2/3}{(x-1)^2} + \frac{-4/9}{x+2}\right) dx \\ &= \frac{4}{9}\ln|x-1| + \frac{2}{3}\cdot\frac{-1}{x-1} - \frac{4}{9}\ln|x+2| + C \\ &= \frac{4}{9}\ln|x-1| - \frac{2}{3(x-1)} - \frac{4}{9}\ln|x+2| + C\end{aligned}$$

Answer: $\boxed{\frac{4}{9}\ln\left|\frac{x-1}{x+2}\right| - \frac{2}{3(x-1)} + C}$

Example 27: Irreducible Quadratic Factor

Problem: Evaluate $\int \frac{x^2+2x+3}{(x+1)(x^2+1)}\,dx$

Solution:

Step 1: Set up partial fractions (x^2+1 is irreducible)

$$\frac{x^2+2x+3}{(x+1)(x^2+1)} = \frac{A}{x+1} + \frac{Bx+C}{x^2+1}$$

Step 2: Multiply by $(x+1)(x^2+1)$

$$x^2 + 2x + 3 = A(x^2+1) + (Bx+C)(x+1)$$

Step 3: Solve for constants

Let $x = -1$: $1 - 2 + 3 = A(2) + 0 \implies 2 = 2A \implies A = 1$

Expand right side:

$$\begin{aligned} x^2+2x+3 &= Ax^2+A+Bx^2+Bx+Cx+C \\ &= (A+B)x^2+(B+C)x+(A+C) \end{aligned}$$

Comparing coefficients:

$$\begin{aligned} x^2: &\quad 1 = A+B \implies B=0 \\ x^1: &\quad 2 = B+C \implies C=2 \\ x^0: &\quad 3 = A+C \implies 3=1+2 \quad \checkmark \end{aligned}$$

So: $A=1$, $B=0$, $C=2$

Step 4: Integrate

$$\begin{aligned} \int \frac{x^2+2x+3}{(x+1)(x^2+1)}\,dx &= \int \left(\frac{1}{x+1}+\frac{2}{x^2+1}\right)dx \\ &= \ln|x+1| + 2\arctan x + C \end{aligned}$$

Answer: $\boxed{\ln|x+1| + 2\arctan x + C}$

Note: Remember $\int \frac{1}{x^2+1}\,dx = \arctan x + C$

Example 28: Improper Rational Function

Problem: Evaluate $\displaystyle\int \frac{x^3}{x^2-1}\,dx$

Solution:

Step 1: Check if proper

deg(num) $=3$, deg(den) $=2 \implies$ Improper! Use long division first.

Step 2: Polynomial long division

$$\frac{x^3}{x^2-1} = x + \frac{x}{x^2-1}$$

Verification:

$$x(x^2-1)+x = x^3-x+x = x^3 \quad \checkmark$$

Step 3: Apply partial fractions to remainder

$$\frac{x}{x^2-1} = \frac{x}{(x-1)(x+1)} = \frac{A}{x-1}+\frac{B}{x+1}$$

Multiply by $(x-1)(x+1)$:

$$x = A(x+1)+B(x-1)$$

Let $x=1$: $1=2A \implies A=\frac{1}{2}$

Let $x=-1$: $-1=-2B \implies B=\frac{1}{2}$

Step 4: Integrate

$$\begin{aligned}\int \frac{x^3}{x^2-1}\,dx &= \int \left(x + \frac{1/2}{x-1} + \frac{1/2}{x+1}\right) dx \\ &= \frac{x^2}{2} + \frac{1}{2}\ln|x-1| + \frac{1}{2}\ln|x+1| + C \\ &= \frac{x^2}{2} + \frac{1}{2}\ln|(x-1)(x+1)| + C \\ &= \frac{x^2}{2} + \frac{1}{2}\ln|x^2-1| + C\end{aligned}$$

Answer: $\boxed{\frac{x^2}{2} + \frac{1}{2}\ln|x^2-1| + C}$

Section 5.6 Summary: Partial Fractions

Procedure:

1. Check if proper; if not, do long division first
2. Factor denominator completely
3. Set up partial fraction form based on factor types
4. Multiply both sides by common denominator
5. Solve for constants (substitution or compare coefficients)
6. Integrate each term separately

Partial Fraction Forms:

Factor Type	Partial Fraction
$(x-a)$ distinct	$\frac{A}{x-a}$
$(x-a)^k$ repeated	$\frac{A_1}{x-a} + \frac{A_2}{(x-a)^2} + \cdots + \frac{A_k}{(x-a)^k}$
ax^2+bx+c irreducible	$\frac{Ax+B}{ax^2+bx+c}$

Common Integrals from Partial Fractions:

$$\begin{aligned}\int \frac{1}{x-a}\,dx &= \ln|x-a| + C \\ \int \frac{1}{(x-a)^n}\,dx &= \frac{-1}{(n-1)(x-a)^{n-1}} + C \quad (n \neq 1) \\ \int \frac{1}{x^2+1}\,dx &= \arctan x + C \\ \int \frac{1}{x^2+a^2}\,dx &= \frac{1}{a}\arctan\left(\frac{x}{a}\right) + C\end{aligned}$$

Common Mistakes:

- Forgetting to do long division for improper fractions
- Wrong partial fraction form for repeated factors

- Using $\frac{A}{ax^2+bx+c}$ instead of $\frac{Ax+B}{ax^2+bx+c}$ for quadratics
- Arithmetic errors when solving for constants
- Forgetting absolute value in logarithms

Pro Tip: Always verify your partial fraction decomposition by combining fractions back together!

Riemann Sums

Riemann sums provide the foundation for understanding definite integrals. They represent the idea of approximating the area under a curve by summing areas of rectangles.

Definition 5.8: Partition of an Interval

A **partition** of the interval $[a, b]$ is a finite set of points:

$$P = \{x_0, x_1, x_2, \dots, x_n\}$$

where:

$$a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b$$

This divides $[a, b]$ into n subintervals:

$$[x_0, x_1], \quad [x_1, x_2], \quad \dots, \quad [x_{n-1}, x_n]$$

Width of i-th subinterval:

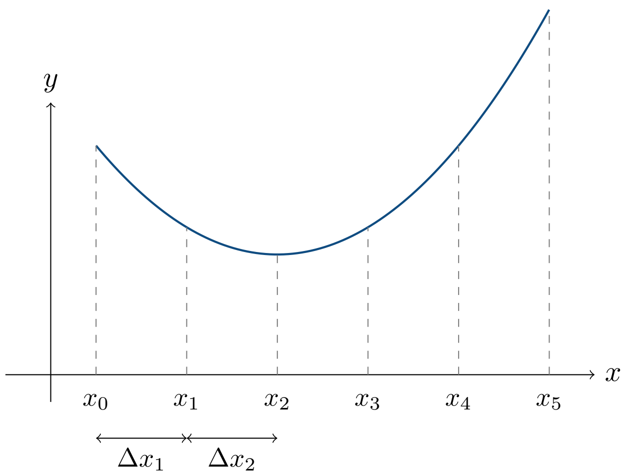
$$\Delta x_i = x_i - x_{i-1}$$

Regular Partition: All subintervals have equal width:

$$\Delta x = \frac{b-a}{n}$$

Sample Points:
In each subinterval $[x_{i-1}, x_i]$, choose a sample point x_i^* where $x_{i-1} \le x_i^* \le x_i$. Common choices:

- **Left endpoint:** $x_i^* = x_{i-1}$
- **Right endpoint:** $x_i^* = x_i$
- **Midpoint:** $x_i^* = \frac{x_{i-1}+x_i}{2}$



Partition of $[a, b]$ into n subintervals

Definition 5.9: Riemann Sum

Let f be defined on $[a, b]$, and let P be a partition with sample points x_i^* in each subinterval. The **Riemann sum** is:

$$\boxed{R_n = \sum_{i=1}^{n} f(x_i^*)\Delta x_i}$$

For a regular partition with $\Delta x = \frac{b-a}{n}$:

$$\boxed{R_n = \sum_{i=1}^{n} f(x_i^*) \cdot \frac{b-a}{n}}$$

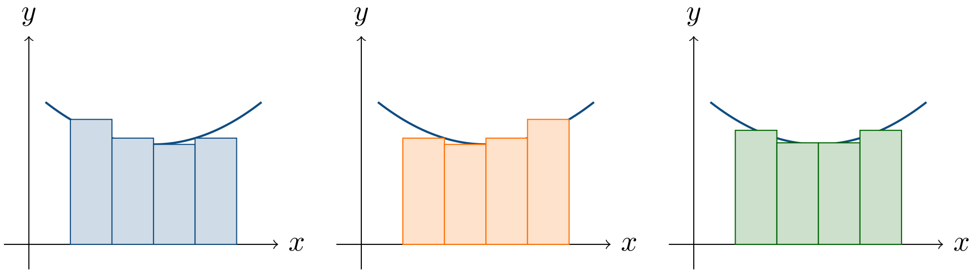
Geometric Interpretation:

The Riemann sum represents the sum of areas of rectangles:

- Height of i-th rectangle: $f(x_i^*)$
- Width of i-th rectangle: Δx_i
- Area of i-th rectangle: $f(x_i^*)\Delta x_i$

Special Cases:

- **Left Riemann Sum:** Use left endpoints $x_i^* = x_{i-1}$
- **Right Riemann Sum:** Use right endpoints $x_i^* = x_i$
- **Midpoint Riemann Sum:** Use midpoints $x_i^* = \frac{x_{i-1}+x_i}{2}$



Left Riemann Sum **Right** Riemann Sum **Midpoint** Riemann Sum

Example 29: Computing a Left Riemann Sum

Problem: Approximate $\int_0^2 x^2\,dx$ using a left Riemann sum with $n = 4$ rectangles.

Solution:

Step 1: Find Δx

$$\Delta x = \frac{b-a}{n} = \frac{2-0}{4} = 0.5$$

Step 2: Find partition points

$$x_0 = 0$$
$$x_1 = 0.5$$
$$x_2 = 1.0$$
$$x_3 = 1.5$$
$$x_4 = 2.0$$

Step 3: Left endpoints are sample points

$$x_1^* = x_0 = 0$$
$$x_2^* = x_1 = 0.5$$
$$x_3^* = x_2 = 1.0$$
$$x_4^* = x_3 = 1.5$$

Step 4: Compute Riemann sum

$$\begin{aligned}
L_4 &= \sum_{i=1}^{4} f(x_i^*)\Delta x \\
&= f(0)(0.5) + f(0.5)(0.5) + f(1.0)(0.5) + f(1.5)(0.5) \\
&= (0^2)(0.5) + (0.5^2)(0.5) + (1^2)(0.5) + (1.5^2)(0.5) \\
&= 0 + 0.125 + 0.5 + 1.125 \\
&= 1.75
\end{aligned}$$

Answer: $\boxed{L_4 = 1.75}$

Note: The exact value is $\int_0^2 x^2\,dx = \frac{8}{3} \approx 2.667$, so the left sum underestimates (as expected for increasing function).

Example 30: Right Riemann Sum

Problem: Approximate $\displaystyle\int_1^3 \frac{1}{x}\,dx$ using a right Riemann sum with $n = 4$.

Solution:

Step 1: Find Δx

$$\Delta x = \frac{3-1}{4} = 0.5$$

Step 2: Partition points

$x_0 = 1$, $x_1 = 1.5$, $x_2 = 2$, $x_3 = 2.5$, $x_4 = 3$

Step 3: Right endpoints as sample points

$x_1^* = 1.5$, $x_2^* = 2$, $x_3^* = 2.5$, $x_4^* = 3$

Step 4: Compute

$$\begin{aligned}
R_4 &= f(1.5)(0.5) + f(2)(0.5) + f(2.5)(0.5) + f(3)(0.5) \\
&= \frac{1}{1.5}(0.5) + \frac{1}{2}(0.5) + \frac{1}{2.5}(0.5) + \frac{1}{3}(0.5) \\
&= 0.5\left(\frac{1}{1.5} + \frac{1}{2} + \frac{1}{2.5} + \frac{1}{3}\right) \\
&= 0.5(0.6667 + 0.5 + 0.4 + 0.3333) \\
&= 0.5(1.9) \\
&= 0.95
\end{aligned}$$

Answer: $\boxed{R_4 \approx 0.95}$

Note: Exact value is $\ln 3 \approx 1.099$, so right sum underestimates (decreasing function).

Theorem 5.8: Limit of Riemann Sums

If f is continuous on $[a, b]$, then:

$$\boxed{\lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x = \int_a^b f(x)\, dx}$$

This limit exists and is independent of the choice of sample points x_i^*.

Meaning:

As we use more and more rectangles ($n \to \infty$), the Riemann sum approaches the exact area under the curve, which we define as the definite integral.

Convergence:

- Left, right, and midpoint sums all converge to the same value
- The integral is defined as this common limit
- Continuous functions are integrable (the limit exists)

Section 5.7 Summary: Riemann Sums

Riemann Sum Formula:

$$\boxed{R_n = \sum_{i=1}^{n} f(x_i^*)\Delta x}$$

For regular partition: $\Delta x = \dfrac{b-a}{n}$

Types of Riemann Sums:

- **Left:** $x_i^* = x_{i-1}$ (left endpoint)
- **Right:** $x_i^* = x_i$ (right endpoint)
- **Midpoint:** $x_i^* = \frac{x_{i-1}+x_i}{2}$

Key Concept:

$$\int_a^b f(x)\, dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x$$

Approximation Quality:

- Increasing function: Left underestimates, Right overestimates
- Decreasing function: Left overestimates, Right underestimates
- Midpoint sum usually gives better approximation than left or right

Understanding: Riemann sums provide the conceptual foundation for definite integrals – they formalize the idea of "area under a curve."

The Definite Integral

The definite integral formalizes the concept of the area under a curve and is defined as the limit of Riemann sums. Unlike indefinite integrals, definite integrals produce a number, not a function.

Definition 5.10: The Definite Integral

Let f be a function defined on $[a, b]$. The **definite integral** of f from a to b is:

$$\boxed{\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x}$$

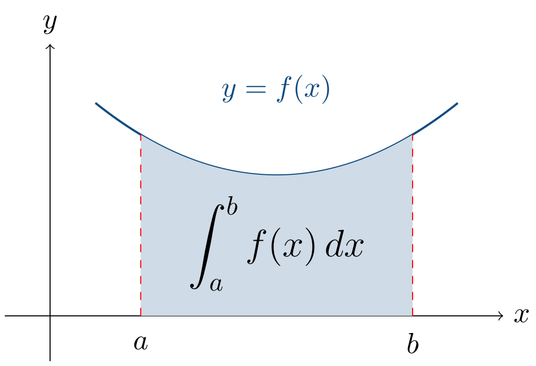
provided this limit exists and is the same for all choices of sample points x_i^*.
Terminology:

- $\int$ is the **integral sign**
- $f(x)$ is the **integrand**
- a is the **lower limit** of integration
- b is the **upper limit** of integration
- dx is the **differential**

When the Limit Exists:
If f is continuous on $[a, b]$, the limit always exists, and we say f is **integrable** on $[a, b]$.

Key Difference:

	Indefinite Integral	Definite Integral
Notation	$\int f(x)\,dx$	$\int_a^b f(x)\,dx$
Result	Function $+C$	Number
Limits	No limits	Has a and b
$+C$	Always included	Never included



Definite Integral = Signed Area

Theorem 5.9: Properties of the Definite Integral

Let f and g be integrable on $[a, b]$, and let c be a constant.

1. Constant Multiple:

$$\int_a^b cf(x)\,dx = c\int_a^b f(x)\,dx$$

2. Sum and Difference:

$$\int_a^b [f(x) \pm g(x)]\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$$

3. Interval Additivity:

If $a < c < b$:

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

4. Zero Width:

$$\int_a^a f(x)\,dx = 0$$

5. Reversal of Limits:

$$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$$

6. Comparison:

If $f(x) \geq g(x)$ for all $x \in [a, b]$:

$$\int_a^b f(x)\,dx \geq \int_a^b g(x)\,dx$$

7. Bounds:

If $m \leq f(x) \leq M$ for all $x \in [a, b]$:

$$m(b-a) \leq \int_a^b f(x)\,dx \leq M(b-a)$$

Example 31: Using Properties of Definite Integrals

Problem: Given $\int_0^2 f(x)\,dx = 5$ and $\int_0^2 g(x)\,dx = -3$, evaluate:

(a) $\int_0^2 [2f(x) + 3g(x)]\,dx$

(b) $\int_2^0 f(x)\,dx$

Solution:

(a) Use linearity properties

$$\begin{aligned}\int_0^2 [2f(x) + 3g(x)]\,dx &= \int_0^2 2f(x)\,dx + \int_0^2 3g(x)\,dx \\ &= 2\int_0^2 f(x)\,dx + 3\int_0^2 g(x)\,dx \\ &= 2(5) + 3(-3) \\ &= 10 - 9 \\ &= 1\end{aligned}$$

Answer (a): $\boxed{1}$

(b) Use reversal of limits

$$\begin{aligned}\int_2^0 f(x)\,dx &= -\int_0^2 f(x)\,dx \\ &= -5\end{aligned}$$

Answer (b): $\boxed{-5}$

Example 32: Additivity Property

Problem: Given $\displaystyle\int_0^4 f(x)\,dx = 10$ and $\displaystyle\int_0^1 f(x)\,dx = 2$, find $\displaystyle\int_1^4 f(x)\,dx$.

Solution:
Use interval additivity with $a = 0$, $c = 1$, $b = 4$:

$$\begin{aligned}\int_0^4 f(x)\,dx &= \int_0^1 f(x)\,dx + \int_1^4 f(x)\,dx \\ 10 &= 2 + \int_1^4 f(x)\,dx \\ \int_1^4 f(x)\,dx &= 8\end{aligned}$$

Answer: $\boxed{8}$

Definition 5.11: Signed Area Interpretation

The definite integral $\int_a^b f(x)\,dx$ represents the **net signed area** between the curve $y = f(x)$ and the x-axis:

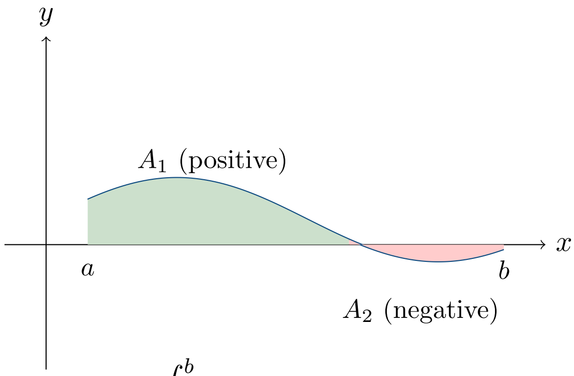
- Area **above** the x-axis: **positive** contribution
- Area **below** the x-axis: **negative** contribution

$$\int_a^b f(x)\,dx = A_1 - A_2$$

where A_1 is area above axis and A_2 is area below axis.

⚠ **Important:**

The definite integral can be negative! It's not always "area" in the geometric sense.



$$\int_a^b f(x)\,dx = A_1 - A_2$$

Example 33: Geometric Interpretation

Problem: Evaluate $\displaystyle\int_{-2}^{2}(4-x^2)\,dx$ using geometry.

Solution:

Step 1: Recognize the shape

$y = 4 - x^2$ is a parabola opening downward with vertex at $(0, 4)$.

Step 2: Find where it crosses the x-axis

$$\begin{aligned} 4 - x^2 &= 0 \\ x^2 &= 4 \\ x &= \pm 2 \end{aligned}$$

So the curve crosses at $x = -2$ and $x = 2$ (exactly our limits!)

Step 3: Recognize symmetry

The parabola is symmetric about the y-axis, and our interval is $[-2, 2]$ (symmetric).

By symmetry:

$$\int_{-2}^{2}(4-x^2)\,dx = 2\int_0^2 (4-x^2)\,dx$$

Step 4: The region from 0 to 2 is bounded by the parabola and x-axis.

This is not a simple geometric shape, so we'll need calculus (Fundamental Theorem) to evaluate exactly.

However, we know the integral is **positive** since $4 - x^2 > 0$ on $(-2, 2)$.

Answer: Using FTC (next section), the exact answer is $\boxed{\dfrac{32}{3}}$.

Geometric Insight: The integral represents the area between the parabola and the x-axis from -2 to 2.

Section 5.8 Summary: The Definite Integral

Definition:

$$\boxed{\int_a^b f(x)\,dx = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i^*)\Delta x}$$

Key Properties:

$$\int_a^b cf(x)\,dx = c\int_a^b f(x)\,dx$$
$$\int_a^b [f \pm g]\,dx = \int_a^b f\,dx \pm \int_a^b g\,dx$$
$$\int_a^b f\,dx = \int_a^c f\,dx + \int_c^b f\,dx$$
$$\int_a^b f\,dx = -\int_b^a f\,dx$$
$$\int_a^a f\,dx = 0$$

Interpretation:

- Definite integral = Net signed area
- Above x-axis: positive
- Below x-axis: negative
- Result is a **number**, not a function

Key Differences from Indefinite Integrals:

Indefinite	Definite
$\int f(x)\,dx$	$\int_a^b f(x)\,dx$
Function $+C$	Number
No limits	Has limits a, b
Antiderivative	Area/Net change

Coming Next: The Fundamental Theorem of Calculus will connect definite integrals to antiderivatives, giving us a powerful way to evaluate definite integrals without computing limits of Riemann sums!

Fundamental Theorem of Calculus

The Fundamental Theorem of Calculus (FTC) is one of the most important theorems in mathematics. It establishes the deep connection between differentiation and integration, showing that they are inverse operations.

Theorem 5.10: The Fundamental Theorem of Calculus – Part 1

If f is continuous on $[a, b]$, then the function g defined by:

$$g(x) = \int_a^x f(t)\,dt \quad \text{for } a \leq x \leq b$$

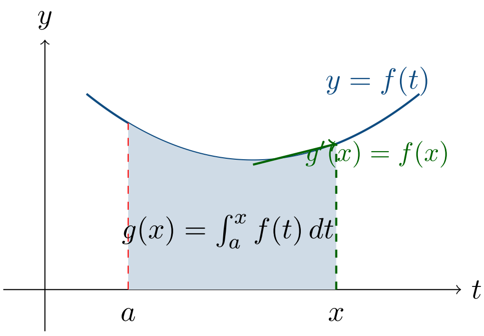
is continuous on $[a, b]$, differentiable on (a, b), and:

$$\boxed{g'(x) = \frac{d}{dx}\left[\int_a^x f(t)\,dt\right] = f(x)}$$

In Words:
The derivative of an integral (with variable upper limit) is the original function.

Meaning:
If you integrate a function and then differentiate the result, you get back the original function. This shows that differentiation and integration are inverse operations!

Note the Variable:
Inside the integral, we use t (dummy variable) instead of x because x is the upper limit.



FTC Part 1: Derivative of Area Function

Example 34: FTC Part 1 – Basic Application

Problem: Find $\frac{d}{dx}\left[\int_1^x t^3\,dt\right]$

Solution:
Using FTC Part 1 with $f(t) = t^3$:

$$\frac{d}{dx}\left[\int_1^x t^3\,dt\right] = x^3$$

Answer: $\boxed{x^3}$

Verification: We can verify by computing the integral first:

$$\int_1^x t^3\,dt = \left[\frac{t^4}{4}\right]_1^x = \frac{x^4}{4} - \frac{1}{4}$$

$$\frac{d}{dx}\left[\frac{x^4}{4} - \frac{1}{4}\right] = x^3 \quad \checkmark$$

Example 35: FTC Part 1 with Chain Rule

Problem: Find $\frac{d}{dx}\left[\int_0^{x^2} \sqrt{1+t^2}\,dt\right]$

Solution:
The upper limit is x^2 (not just x), so we need the chain rule.
Let $u = x^2$. Then:

$$\frac{d}{dx}\left[\int_0^u \sqrt{1+t^2}\,dt\right] = \sqrt{1+u^2} \cdot \frac{du}{dx}$$

Step 1: Apply FTC Part 1

$$\frac{d}{du}\left[\int_0^u \sqrt{1+t^2}\,dt\right] = \sqrt{1+u^2}$$

Step 2: Apply chain rule

$$\frac{d}{dx} = \frac{d}{du} \cdot \frac{du}{dx} = \sqrt{1+u^2} \cdot 2x$$

Step 3: Substitute back $u = x^2$

$$= \sqrt{1+(x^2)^2} \cdot 2x = 2x\sqrt{1+x^4}$$

Answer: $\boxed{2x\sqrt{1+x^4}}$

General Formula:

$$\frac{d}{dx}\left[\int_a^{g(x)} f(t)\,dt\right] = f(g(x)) \cdot g'(x)$$

Theorem 5.11: The Fundamental Theorem of Calculus – Part 2

If f is continuous on $[a,b]$ and F is any antiderivative of f (meaning $F' = f$), then:

$$\boxed{\int_a^b f(x)\,dx = F(b) - F(a)}$$

Notation:

$$\int_a^b f(x)\,dx = [F(x)]_a^b = F(b) - F(a)$$

In Words:
To evaluate a definite integral, find any antiderivative and compute the difference of its values at the endpoints.

Why This is Powerful:
Instead of computing limits of Riemann sums, we can:

1. Find an antiderivative F
2. Evaluate at endpoints: $F(b) - F(a)$

This makes evaluating definite integrals much easier!

Important:
We don't need $+C$ when evaluating definite integrals because:

$$[F(x) + C]_a^b = [F(b) + C] - [F(a) + C] = F(b) - F(a)$$

The constant cancels out!

Example 36: FTC Part 2 – Basic Evaluation

Problem: Evaluate $\int_1^3 x^2\,dx$

Solution:
Step 1: Find an antiderivative
An antiderivative of $f(x) = x^2$ is $F(x) = \frac{x^3}{3}$.
Step 2: Apply FTC Part 2

$$\begin{aligned}\int_1^3 x^2\,dx &= \left[\frac{x^3}{3}\right]_1^3 \\ &= \frac{3^3}{3} - \frac{1^3}{3} \\ &= \frac{27}{3} - \frac{1}{3} \\ &= 9 - \frac{1}{3} \\ &= \frac{26}{3}\end{aligned}$$

Answer: $\boxed{\frac{26}{3}}$
Interpretation: The area under $y = x^2$ from $x = 1$ to $x = 3$ is $\frac{26}{3}$ square units.

Example 37: Definite Integral with Negative Region

Problem: Evaluate $\int_0^\pi \sin x\,dx$

Solution:
Step 1: Find antiderivative
An antiderivative of $\sin x$ is $-\cos x$.
Step 2: Apply FTC

$$\begin{aligned}\int_0^{\pi} \sin x\,dx &= [-\cos x]_0^{\pi} \\ &= -\cos(\pi) - (-\cos(0)) \\ &= -(-1) - (-1) \\ &= 1 + 1 \\ &= 2\end{aligned}$$

Answer: $\boxed{2}$

Note: $\sin x \geq 0$ on $[0, \pi]$, so this represents actual area (positive).

Example 38: Definite Integral of Polynomial

Problem: Evaluate $\int_{-1}^{2} (3x^2 - 2x + 1)\,dx$

Solution:

Step 1: Find antiderivative

$$\begin{aligned}F(x) &= \int (3x^2 - 2x + 1)\,dx \\ &= x^3 - x^2 + x\end{aligned}$$

Step 2: Apply FTC

$$\begin{aligned}\int_{-1}^{2} (3x^2 - 2x + 1)\,dx &= [x^3 - x^2 + x]_{-1}^{2} \\ &= [2^3 - 2^2 + 2] - [(-1)^3 - (-1)^2 + (-1)] \\ &= [8 - 4 + 2] - [-1 - 1 - 1] \\ &= 6 - (-3) \\ &= 9\end{aligned}$$

Answer: $\boxed{9}$

Example 39: Using Substitution with Definite Integrals

Problem: Evaluate $\int_0^1 x\sqrt{1 + x^2}\,dx$

Solution:

Method 1: Change limits with substitution

Let $u = 1 + x^2$, then $du = 2x\,dx$, so $x\,dx = \frac{1}{2}du$

Change limits:

- When $x = 0$: $u = 1 + 0^2 = 1$
- When $x = 1$: $u = 1 + 1^2 = 2$

$$
\begin{aligned}
\int_0^1 x\sqrt{1+x^2}\,dx &= \int_1^2 \sqrt{u}\cdot\frac{1}{2}\,du \\
&= \frac{1}{2}\int_1^2 u^{1/2}\,du \\
&= \frac{1}{2}\left[\frac{u^{3/2}}{3/2}\right]_1^2 \\
&= \frac{1}{2}\cdot\frac{2}{3}\left[u^{3/2}\right]_1^2 \\
&= \frac{1}{3}\left[2^{3/2}-1^{3/2}\right] \\
&= \frac{1}{3}\left[2\sqrt{2}-1\right] \\
&= \frac{2\sqrt{2}-1}{3}
\end{aligned}
$$

Method 2: Find antiderivative first, then substitute back

$$\int x\sqrt{1+x^2}\,dx = \frac{1}{3}(1+x^2)^{3/2}+C$$

Evaluate:

$$
\begin{aligned}
\left[\frac{1}{3}(1+x^2)^{3/2}\right]_0^1 &= \frac{1}{3}(2)^{3/2}-\frac{1}{3}(1)^{3/2} \\
&= \frac{2\sqrt{2}-1}{3}
\end{aligned}
$$

Answer: $\boxed{\dfrac{2\sqrt{2}-1}{3}}$

Note: Both methods work! Method 1 is often cleaner for definite integrals.

Example 40: Net Change Interpretation

Problem: A particle moves along a line with velocity $v(t) = 3t^2 - 6t$ m/s. Find the displacement from $t = 0$ to $t = 3$ seconds.

Solution:
Concept: Displacement $= \int$ velocity

$$
\begin{aligned}
\text{Displacement} &= \int_0^3 v(t)\,dt \\
&= \int_0^3 (3t^2-6t)\,dt \\
&= [t^3-3t^2]_0^3 \\
&= [3^3-3(3^2)]-[0-0] \\
&= [27-27]-0 \\
&= 0
\end{aligned}
$$

Answer: $\boxed{0 \text{ meters}}$

Interpretation: The particle returns to its starting position! The definite integral gives the **net displacement**, not total distance traveled.

Note: To find total distance, we'd need to integrate $|v(t)|$ (considering when velocity is negative).

Section 5.9 Summary: Fundamental Theorem of Calculus

FTC Part 1:

$$\boxed{\frac{d}{dx}\left[\int_a^x f(t)\,dt\right] = f(x)}$$

Differentiation and integration are inverse operations.

FTC Part 2 (Evaluation Theorem):

$$\boxed{\int_a^b f(x)\,dx = F(b) - F(a)}$$

where F is any antiderivative of f.

Procedure for Evaluating Definite Integrals:

1. Find an antiderivative $F(x)$ of $f(x)$
2. Evaluate $F(b) - F(a)$
3. No need for $+C$ (it cancels)

With Substitution:

- **Option 1:** Change the limits when you substitute
- **Option 2:** Find antiderivative, substitute back, then evaluate

Net Change Theorem:

If $F' = f$, then:

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

represents the net change in F from a to b.

Common Mistakes:

- Including $+C$ in definite integrals (unnecessary)
- Forgetting to change limits when substituting
- Computing $F(a) - F(b)$ instead of $F(b) - F(a)$
- Confusing FTC Part 1 and Part 2

Exam Alert: FTC is fundamental! Appears in almost every calculus exam.

Area Under Curves and Between Curves

One of the most important applications of definite integrals is finding areas of regions in the plane. This includes areas under curves and areas between two curves.

Definition 5.12: Area Under a Curve

If f is continuous and $f(x) \geq 0$ on $[a, b]$, then the area A of the region bounded by:

- The curve $y = f(x)$
- The x-axis
- The vertical lines $x = a$ and $x = b$

is given by:

$$\boxed{A = \int_a^b f(x)\,dx}$$

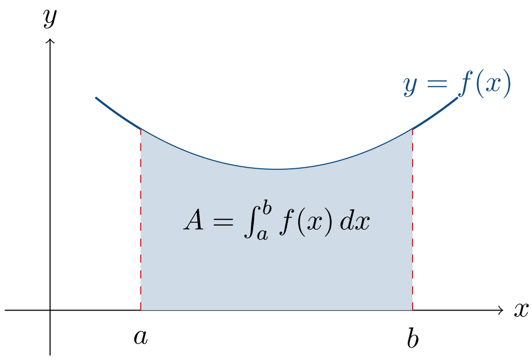
If $f(x) < 0$:

When the curve is below the x-axis, the integral is negative. To find the actual area (always positive), use:

$$A = \int_a^b |f(x)|\,dx = -\int_a^b f(x)\,dx \quad (\text{when } f(x) \leq 0)$$

Mixed Case:

If $f(x)$ is sometimes positive and sometimes negative, split the integral at the zeros.



Area Under Curve

Example 41: Area Under a Curve

Problem: Find the area of the region bounded by $y = x^2$, the x-axis, and the lines $x = 1$ and $x = 3$.

Solution:

Step 1: Set up the integral

Since $x^2 \geq 0$ on $[1, 3]$:

$$A = \int_1^3 x^2\,dx$$

Step 2: Evaluate

$$A = \left[\frac{x^3}{3}\right]_1^3 = \frac{27}{3} - \frac{1}{3} = \frac{26}{3}$$

Answer: $\boxed{A = \frac{26}{3} \text{ square units}}$

Theorem 5.12: Area Between Two Curves

If f and g are continuous on $[a, b]$ and $f(x) \geq g(x)$ for all $x \in [a, b]$, then the area A of the region between the curves is:

$$\boxed{A = \int_a^b [f(x) - g(x)]\, dx}$$

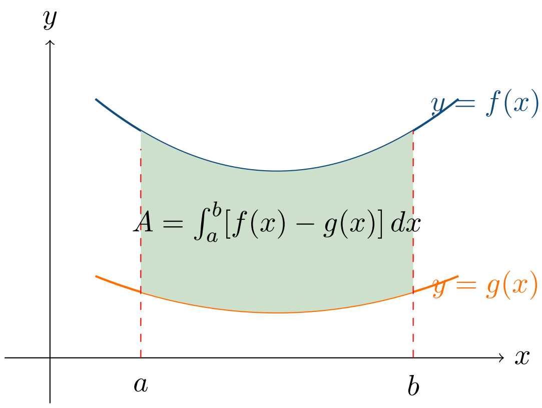
Think of it as:

$$A = \int_a^b [\text{top} - \text{bottom}]\, dx$$

Procedure:

1. Sketch the curves
2. Identify which is on top: $f(x) \geq g(x)$
3. Find intersection points (if needed)
4. Set up: $A = \int_a^b [f(x) - g(x)]\, dx$
5. Evaluate the integral

Critical:
Always subtract bottom from top! If you integrate $g(x) - f(x)$ when $f(x) > g(x)$, you'll get a negative area.



Area Between Two Curves

Example 42: Area Between Two Curves

Problem: Find the area of the region bounded by $y = x^2$ and $y = 2x + 3$.

Solution:
Step 1: Find intersection points
Set $x^2 = 2x + 3$:

$$x^2 - 2x - 3 = 0$$
$$(x-3)(x+1) = 0$$
$$x = -1 \text{ or } x = 3$$

Step 2: Determine which is on top
Test a point between -1 and 3, say $x = 0$:

- $y = x^2 = 0$
- $y = 2x + 3 = 3$

So $2x + 3 > x^2$ on $[-1, 3]$.
Step 3: Set up integral

$$A = \int_{-1}^{3} [(2x+3) - x^2]\,dx$$

Step 4: Evaluate

$$
\begin{aligned}
A &= \int_{-1}^{3} (2x + 3 - x^2)\, dx \\
&= \left[x^2 + 3x - \frac{x^3}{3}\right]_{-1}^{3} \\
&= [9 + 9 - 9] - \left[1 - 3 + \frac{1}{3}\right] \\
&= 9 - \left(-2 + \frac{1}{3}\right) \\
&= 9 + 2 - \frac{1}{3} \\
&= 11 - \frac{1}{3} \\
&= \frac{32}{3}
\end{aligned}
$$

Answer: $\boxed{A = \frac{32}{3} \text{ square units}}$

Example 43: Area Enclosed by Two Curves

Problem: Find the area enclosed by $y = \sin x$ and $y = \cos x$ from $x = 0$ to their first intersection point.

Solution:
Step 1: Find intersection

$$
\begin{aligned}
\sin x &= \cos x \\
\tan x &= 1 \\
x &= \frac{\pi}{4} \quad \text{(first intersection)}
\end{aligned}
$$

Step 2: Determine which is on top on $[0, \pi/4]$
At $x = 0$: $\cos 0 = 1 > \sin 0 = 0$
So $\cos x > \sin x$ on $[0, \pi/4]$.
Step 3: Set up and evaluate

$$
\begin{aligned}
A &= \int_{0}^{\pi/4} (\cos x - \sin x)\, dx \\
&= [\sin x + \cos x]_0^{\pi/4} \\
&= \left[\sin\frac{\pi}{4} + \cos\frac{\pi}{4}\right] - [\sin 0 + \cos 0] \\
&= \left[\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\right] - [0 + 1] \\
&= \sqrt{2} - 1
\end{aligned}
$$

Answer: $\boxed{A = \sqrt{2} - 1 \text{ square units}}$

Example 44: Area with Curve Below x-axis

Problem: Find the area of the region bounded by $y = x^2 - 4$ and the x-axis from $x = -2$ to $x = 2$.

Solution:
Step 1: Observe that $x^2 - 4 \leq 0$ on $[-2, 2]$
The curve is below the x-axis.
Step 2: For actual area, use $|f(x)|$ or $-f(x)$

$$\begin{aligned} A &= \int_{-2}^{2} |x^2 - 4|\, dx \\ &= \int_{-2}^{2} -(x^2 - 4)\, dx \\ &= \int_{-2}^{2} (4 - x^2)\, dx \end{aligned}$$

Step 3: Evaluate

$$\begin{aligned} A &= \left[4x - \frac{x^3}{3}\right]_{-2}^{2} \\ &= \left[8 - \frac{8}{3}\right] - \left[-8 + \frac{8}{3}\right] \\ &= 8 - \frac{8}{3} + 8 - \frac{8}{3} \\ &= 16 - \frac{16}{3} \\ &= \frac{32}{3} \end{aligned}$$

Answer: $\boxed{A = \frac{32}{3} \text{ square units}}$

Note: If we had computed $\int_{-2}^{2} (x^2 - 4)\, dx$, we'd get $-\frac{32}{3}$ (negative because curve is below axis). The area is always positive!

Section 5.10 Summary: Area Applications

Area Under a Curve:
If $f(x) \geq 0$ on $[a, b]$:

$$\boxed{A = \int_a^b f(x)\, dx}$$

If $f(x) \leq 0$ on $[a, b]$:

$$\boxed{A = -\int_a^b f(x)\, dx = \int_a^b |f(x)|\, dx}$$

Area Between Two Curves:

If $f(x) \geq g(x)$ on $[a, b]$:

$$\boxed{A = \int_a^b [f(x) - g(x)]\, dx = \int_a^b [\text{top} - \text{bottom}]\, dx}$$

Step-by-Step Procedure:

1. Sketch both curves
2. Find intersection points (set equal and solve)
3. Determine which curve is on top
4. Set up integral with (top − bottom)
5. Evaluate using FTC

⚠ **Common Mistakes:**

- Subtracting in wrong order (bottom − top gives negative!)
- Forgetting to find intersection points
- Not checking which curve is on top
- Confusing definite integral value (can be negative) with area (always positive)

Exam Alert: Area between curves appears frequently in exams. Always sketch the region first!

Pro Tip: When in doubt, sketch! A quick graph makes it clear which function is on top.

Volumes of Solids of Revolution

When a region in the plane is revolved around an axis, it generates a three-dimensional solid. We can use definite integrals to calculate the volumes of such solids using two main methods: the disk method and the shell method.

Definition 5.13: Solid of Revolution

A **solid of revolution** is a three-dimensional solid obtained by rotating a two-dimensional region about an axis (usually the x-axis or y-axis).

Examples:

- Rotating a rectangle about an axis $\rightarrow$ cylinder
- Rotating a circle about a diameter $\rightarrow$ sphere
- Rotating a triangle about an axis $\rightarrow$ cone

Theorem 5.13: Disk Method (Revolution about x-axis)

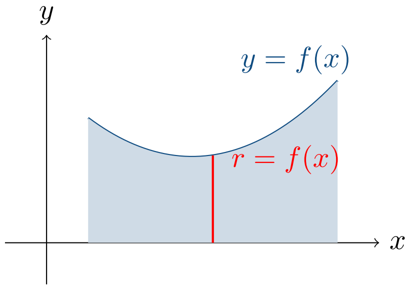
If f is continuous and $f(x) \geq 0$ on $[a, b]$, then the volume of the solid obtained by rotating the region under $y = f(x)$ about the x-axis is:

$$\boxed{V = \int_a^b \pi[f(x)]^2\, dx = \int_a^b \pi y^2\, dx}$$

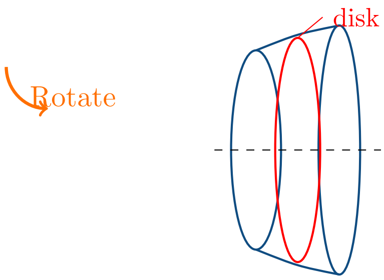
Why This Works:

- Slice the solid perpendicular to the axis of rotation
- Each slice is approximately a disk (circular cross-section)
- Radius of disk at position x is $r = f(x)$
- Area of disk: $A = \pi r^2 = \pi[f(x)]^2$
- Thickness: dx
- Volume of disk: $dV = \pi[f(x)]^2\, dx$
- Sum up all disks: $V = \int_a^b \pi[f(x)]^2\, dx$

Key Point:
Always square the radius: $V = \pi \int r^2\, dx$, not $\pi \int r\, dx$!



Region to Rotate



Solid of Revolution

Example 45: Disk Method – Basic

Problem: Find the volume of the solid obtained by rotating the region bounded by $y = \sqrt{x}$, $y = 0$, $x = 0$, and $x = 4$ about the x-axis.

Solution:
Step 1: Identify the setup
Rotating about x-axis, so use disk method with radius $r = f(x) = \sqrt{x}$.
Step 2: Set up integral

$$V = \int_0^4 \pi[\sqrt{x}]^2\,dx$$
$$= \int_0^4 \pi x\,dx$$

Step 3: Evaluate

$$V = \pi\left[\frac{x^2}{2}\right]_0^4$$
$$= \pi\left(\frac{16}{2} - 0\right)$$
$$= 8\pi$$

Answer: $\boxed{V = 8\pi \text{ cubic units}}$

Theorem 5.14: Washer Method (Hollow Solid)

If the region between two curves $y = f(x)$ and $y = g(x)$ (where $f(x) \geq g(x) \geq 0$) on $[a, b]$ is rotated about the x-axis, the volume is:

$$\boxed{V = \int_a^b \pi\left([f(x)]^2 - [g(x)]^2\right)\,dx}$$

Think of it as:

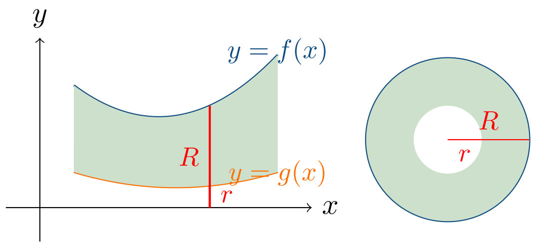
$$V = \int_a^b \pi\left(R^2 - r^2\right)\,dx$$

where:

- $R = f(x)$ is the outer radius
- $r = g(x)$ is the inner radius

Why "Washer"?
Each cross-section is a disk with a hole (like a washer). The area is:

$$A = \pi R^2 - \pi r^2 = \pi(R^2 - r^2)$$



Washer Cross-Section
$A = \pi(R^2 - r^2)$

Example 46: Washer Method

Problem: Find the volume of the solid obtained by rotating the region between $y = x^2$ and $y = x$ about the x-axis.

Solution:
Step 1: Find intersection points

$$\begin{aligned} x^2 &= x \\ x^2 - x &= 0 \\ x(x-1) &= 0 \\ x &= 0 \text{ or } x = 1 \end{aligned}$$

Step 2: Determine which is outer radius
On $(0, 1)$: $x > x^2$, so $R = x$ (outer), $r = x^2$ (inner).
Step 3: Set up integral

$$\begin{aligned} V &= \int_0^1 \pi \left(x^2 - (x^2)^2\right) dx \\ &= \pi \int_0^1 (x^2 - x^4)\, dx \end{aligned}$$

Step 4: Evaluate

$$
\begin{aligned}
V &= \pi \left[\frac{x^3}{3} - \frac{x^5}{5} \right]_0^1 \\
&= \pi \left(\frac{1}{3} - \frac{1}{5} \right) \\
&= \pi \left(\frac{5-3}{15} \right) \\
&= \frac{2\pi}{15}
\end{aligned}
$$

Answer: $\boxed{V = \frac{2\pi}{15} \text{ cubic units}}$

Theorem 5.15: Shell Method (Revolution about y-axis)

If the region bounded by $x = f(y)$, $x = 0$, $y = c$, and $y = d$ is rotated about the y-axis, the volume is:

$$\boxed{V = \int_c^d 2\pi y \cdot f(y)\, dy}$$

More generally, for revolution about the y-axis:

$$\boxed{V = \int_a^b 2\pi x \cdot f(x)\, dx}$$

Shell Method Formula:

$$V = \int 2\pi(\text{radius})(\text{height})\, dx$$

where:

- Radius = distance from axis to shell = x
- Height = value of function = $f(x)$

Why "Shell"?

Each element forms a cylindrical shell:

- Circumference: $2\pi r = 2\pi x$
- Height: $h = f(x)$
- Thickness: dx
- Surface area: $2\pi x \cdot f(x)$
- Volume: $2\pi x \cdot f(x)\, dx$

Example 47: Shell Method

Problem: Find the volume when the region bounded by $y = x^2$, $y = 0$, and $x = 2$ is rotated about the y-axis.

Solution:
Step 1: Identify setup
Rotating about y-axis, so use shell method.
Step 2: Identify radius and height

- Radius: $r = x$ (distance from y-axis)
- Height: $h = x^2$ (value of function)
- Limits: x goes from 0 to 2

Step 3: Set up integral

$$V = \int_0^2 2\pi x \cdot x^2 \, dx = 2\pi \int_0^2 x^3 \, dx$$

Step 4: Evaluate

$$\begin{aligned} V &= 2\pi \left[\frac{x^4}{4}\right]_0^2 \\ &= 2\pi \left(\frac{16}{4} - 0\right) \\ &= 2\pi(4) \\ &= 8\pi \end{aligned}$$

Answer: $\boxed{V = 8\pi \text{ cubic units}}$

Example 48: Choosing Between Methods

Problem: Find the volume when the region between $y = x$ and $y = x^2$ is rotated about:
(a) The x-axis (use washer method)
(b) The y-axis (use shell method)

Solution:
Intersection points: $x = 0$ and $x = 1$ (from Example 46)

(a) About x-axis (Washer Method)

$$\begin{aligned} V &= \int_0^1 \pi(x^2 - (x^2)^2) \, dx \\ &= \frac{2\pi}{15} \quad \text{(from Example 46)} \end{aligned}$$

Answer (a): $\boxed{V = \frac{2\pi}{15}}$

(b) About y-axis (Shell Method)

Radius: $r = x$
Height: $h = x - x^2$ (top minus bottom)

$$\begin{aligned} V &= \int_0^1 2\pi x(x - x^2)\,dx \\ &= 2\pi \int_0^1 (x^2 - x^3)\,dx \\ &= 2\pi \left[\frac{x^3}{3} - \frac{x^4}{4}\right]_0^1 \\ &= 2\pi \left(\frac{1}{3} - \frac{1}{4}\right) \\ &= 2\pi \left(\frac{1}{12}\right) \\ &= \frac{\pi}{6} \end{aligned}$$

Answer (b): $\boxed{V = \frac{\pi}{6}}$

Note: Different axes give different volumes!

Section 5.11 Summary: Volumes of Revolution

Disk Method (about x-axis):

$$\boxed{V = \int_a^b \pi [f(x)]^2\,dx}$$

Washer Method (hollow solid):

$$\boxed{V = \int_a^b \pi \left(R^2 - r^2\right)\,dx}$$

where R = outer radius, r = inner radius.

Shell Method (about y-axis):

$$\boxed{V = \int_a^b 2\pi x \cdot f(x)\,dx}$$

Formula: $V = \int 2\pi(\text{radius})(\text{height})\,dx$

How to Choose:

Rotation Axis	Prefer Disk/Washer	Prefer Shell
x-axis	Function of x	Complex setup
y-axis	Complex setup	Function of x

Common Mistakes:

- Forgetting to square the radius in disk/washer method
- Confusing outer and inner radii in washer method

- Using wrong formula for axis of rotation
- Forgetting the 2π in shell method
- Not finding intersection points first

Pro Tip: Always sketch the region and identify radii/heights before setting up the integral!

Arc Length and Surface Area

Beyond areas and volumes, we can use integration to find the length of curves (arc length) and the surface area of solids of revolution.

Theorem 5.16: Arc Length Formula

If f' is continuous on $[a, b]$, then the length L of the curve $y = f(x)$ from $x = a$ to $x = b$ is:

$$\boxed{L = \int_a^b \sqrt{1 + [f'(x)]^2}\, dx}$$

Alternative notation:

$$\boxed{L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx}$$

Intuition:

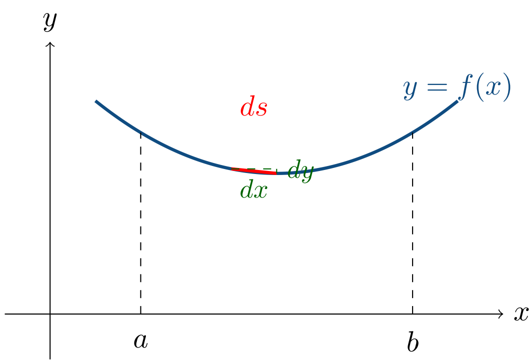
For a small segment of curve:

- Horizontal distance: dx
- Vertical distance: $dy = f'(x)\, dx$
- Arc length element: $ds = \sqrt{(dx)^2 + (dy)^2}$

$$\begin{aligned} ds &= \sqrt{(dx)^2 + (dy)^2} \\ &= \sqrt{(dx)^2 + [f'(x)]^2 (dx)^2} \\ &= \sqrt{1 + [f'(x)]^2}\, dx \end{aligned}$$

Warning:

Arc length integrals are often difficult or impossible to evaluate in closed form!



$$ds = \sqrt{(dx)^2 + (dy)^2} = \sqrt{1 + (f')^2}\, dx$$

Example 49: Arc Length – Simple Case

Problem: Find the length of the curve $y = \frac{2}{3}x^{3/2}$ from $x = 0$ to $x = 3$.

Solution:

Step 1: Find $\frac{dy}{dx}$

$$\frac{dy}{dx} = \frac{2}{3} \cdot \frac{3}{2}x^{1/2} = x^{1/2} = \sqrt{x}$$

Step 2: Set up arc length integral

$$\begin{aligned} L &= \int_0^3 \sqrt{1 + (\sqrt{x})^2}\, dx \\ &= \int_0^3 \sqrt{1 + x}\, dx \end{aligned}$$

Step 3: Evaluate using substitution
Let $u = 1 + x$, $du = dx$
When $x = 0$: $u = 1$
When $x = 3$: $u = 4$

$$\begin{aligned} L &= \int_1^4 \sqrt{u}\, du \\ &= \int_1^4 u^{1/2}\, du \\ &= \left[\frac{u^{3/2}}{3/2}\right]_1^4 \\ &= \frac{2}{3}\left[u^{3/2}\right]_1^4 \\ &= \frac{2}{3}(4^{3/2} - 1^{3/2}) \\ &= \frac{2}{3}(8 - 1) \\ &= \frac{14}{3} \end{aligned}$$

Answer: $\boxed{L = \frac{14}{3} \text{ units}}$

Example 50: Arc Length of a Line Segment

Problem: Verify the arc length formula by finding the length of $y = 2x + 1$ from $x = 0$ to $x = 3$.

Solution:

Step 1: Find derivative

$$\frac{dy}{dx} = 2$$

Step 2: Apply arc length formula

$$\begin{aligned} L &= \int_0^3 \sqrt{1+2^2}\,dx \\ &= \int_0^3 \sqrt{5}\,dx \\ &= \sqrt{5}\cdot[x]_0^3 \\ &= 3\sqrt{5} \end{aligned}$$

Step 3: Verify using distance formula
Endpoints: $(0,1)$ and $(3,7)$

$$\begin{aligned} d &= \sqrt{(3-0)^2+(7-1)^2} \\ &= \sqrt{9+36} \\ &= \sqrt{45} \\ &= 3\sqrt{5} \quad \checkmark \end{aligned}$$

Answer: $\boxed{L = 3\sqrt{5}\text{ units}}$
Note: The arc length formula works correctly for straight lines!

Theorem 5.17: Surface Area of Revolution

If f is continuous and $f(x) \geq 0$ on $[a,b]$, and f' is continuous, then the surface area S of the solid obtained by rotating the curve $y = f(x)$ about the x-axis is:

$$\boxed{S = \int_a^b 2\pi f(x)\sqrt{1+[f'(x)]^2}\,dx}$$

Alternative form:

$$\boxed{S = \int_a^b 2\pi y\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx}$$

Formula Structure:

$$S = \int 2\pi(\text{radius})(\text{arc length element})\,dx$$

where:

- Radius $= y = f(x)$ (distance from axis)
- Arc length element $= \sqrt{1+[f'(x)]^2}\,dx$

Intuition:
Think of the surface as composed of thin frustums (truncated cones). The lateral surface area of each frustum is approximately $2\pi r\,ds$ where:

- $r = f(x)$ is the radius
- $ds = \sqrt{1+[f'(x)]^2}\,dx$ is the slant height

Example 51: Surface Area of a Cone

Problem: Find the surface area of a cone with base radius r and height h by rotating $y = \frac{r}{h}x$ about the x-axis from $x = 0$ to $x = h$.

Solution:

Step 1: Find derivative

$$\frac{dy}{dx} = \frac{r}{h}$$

Step 2: Set up surface area integral

$$\begin{aligned} S &= \int_0^h 2\pi \left(\frac{r}{h}x\right) \sqrt{1 + \left(\frac{r}{h}\right)^2}\, dx \\ &= 2\pi \frac{r}{h}\sqrt{1 + \frac{r^2}{h^2}} \int_0^h x\, dx \\ &= 2\pi \frac{r}{h} \cdot \frac{\sqrt{h^2 + r^2}}{h} \int_0^h x\, dx \end{aligned}$$

Step 3: Simplify and evaluate

$$\begin{aligned} S &= \frac{2\pi r\sqrt{h^2 + r^2}}{h^2} \left[\frac{x^2}{2}\right]_0^h \\ &= \frac{2\pi r\sqrt{h^2 + r^2}}{h^2} \cdot \frac{h^2}{2} \\ &= \pi r\sqrt{h^2 + r^2} \end{aligned}$$

Note that $\sqrt{h^2 + r^2} = \ell$ (slant height).

Answer: $\boxed{S = \pi r\ell}$

Verification: This matches the standard geometry formula for lateral surface area of a cone!

Example 52: Surface Area of Revolution

Problem: Find the surface area when $y = x^3$ from $x = 0$ to $x = 1$ is rotated about the x-axis.

Solution:

Step 1: Find derivative

$$\frac{dy}{dx} = 3x^2$$

Step 2: Set up integral

$$\begin{aligned} S &= \int_0^1 2\pi x^3 \sqrt{1 + (3x^2)^2}\, dx \\ &= 2\pi \int_0^1 x^3 \sqrt{1 + 9x^4}\, dx \end{aligned}$$

Step 3: Use substitution

Let $u = 1 + 9x^4$, then $du = 36x^3\,dx$, so $x^3\,dx = \dfrac{du}{36}$

When $x = 0$: $u = 1$

When $x = 1$: $u = 10$

$$\begin{aligned}
S &= 2\pi \int_1^{10} \sqrt{u} \cdot \frac{du}{36} \\
&= \frac{2\pi}{36} \int_1^{10} u^{1/2}\,du \\
&= \frac{\pi}{18} \left[\frac{u^{3/2}}{3/2}\right]_1^{10} \\
&= \frac{\pi}{18} \cdot \frac{2}{3} \left[u^{3/2}\right]_1^{10} \\
&= \frac{\pi}{27} \left[10^{3/2} - 1\right] \\
&= \frac{\pi}{27} \left[10\sqrt{10} - 1\right]
\end{aligned}$$

Answer: $\boxed{S = \dfrac{\pi(10\sqrt{10} - 1)}{27} \text{ square units}}$

Section 5.12 Summary: Arc Length and Surface Area

Arc Length:

$$\boxed{L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx}$$

Surface Area (revolution about x-axis):

$$\boxed{S = \int_a^b 2\pi y \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx}$$

Key Components:

- Arc length element: $ds = \sqrt{1 + (dy/dx)^2}\,dx$
- Surface area $= 2\pi\times$ radius $\times$ arc length element

Common Patterns:

- Arc length integrals often require substitution
- Many cannot be evaluated in closed form (use numerical methods)
- Surface area combines arc length with rotation concepts

Common Mistakes:

- Forgetting the square root in arc length formula
- Forgetting to square $f'(x)$ inside the square root

- Omitting 2π or the radius in surface area formula
- Incorrect substitution in complex integrals

Pro Tip: Always verify formulas by checking simple cases (e.g., straight lines, cones).

Improper Integrals

All the definite integrals we've studied so far have been proper integrals – they have finite limits of integration and the integrand is continuous on the interval. Improper integrals extend integration to handle infinite limits or discontinuous integrands.

Definition 5.14: Types of Improper Integrals

An integral is **improper** if:

Type 1: One or both limits of integration are infinite

- $\int_a^\infty f(x)\,dx$ (infinite upper limit)
- $\int_{-\infty}^b f(x)\,dx$ (infinite lower limit)
- $\int_{-\infty}^\infty f(x)\,dx$ (both limits infinite)

Type 2: The integrand has an infinite discontinuity in $[a,b]$

- f is discontinuous at a, b, or some point in (a,b)
- $\lim_{x\to c} |f(x)| = \infty$ for some $c \in [a,b]$

Key Concept:
We evaluate improper integrals as **limits** of proper integrals.

Theorem 5.18: Improper Integrals – Type 1 (Infinite Limits)

(a) Infinite Upper Limit:

$$\boxed{\int_a^\infty f(x)\,dx = \lim_{t\to\infty} \int_a^t f(x)\,dx}$$

(b) Infinite Lower Limit:

$$\boxed{\int_{-\infty}^b f(x)\,dx = \lim_{t\to-\infty} \int_t^b f(x)\,dx}$$

(c) Both Limits Infinite:

$$\boxed{\int_{-\infty}^\infty f(x)\,dx = \int_{-\infty}^c f(x)\,dx + \int_c^\infty f(x)\,dx}$$

for any real number c. Both integrals on the right must converge.

Convergence vs. Divergence:

- If the limit exists and is finite: the integral **converges**
- If the limit is $\pm\infty$ or doesn't exist: the integral **diverges**

Important:
For case (c), you must split at some point c and both parts must converge independently. You cannot use:

$$\int_{-\infty}^\infty f(x)\,dx \neq \lim_{t\to\infty} \int_{-t}^t f(x)\,dx \quad \text{(WRONG!)}$$

Example 53: Convergent Improper Integral

Problem: Determine whether $\int_1^\infty \frac{1}{x^2}\,dx$ converges or diverges. If it converges, find its value.

Solution:
Step 1: Write as a limit

$$\int_1^\infty \frac{1}{x^2}\,dx = \lim_{t\to\infty} \int_1^t \frac{1}{x^2}\,dx$$

Step 2: Evaluate the proper integral

$$\begin{aligned}\int_1^t \frac{1}{x^2}\,dx &= \int_1^t x^{-2}\,dx \\ &= \left[\frac{x^{-1}}{-1}\right]_1^t \\ &= \left[-\frac{1}{x}\right]_1^t \\ &= -\frac{1}{t} - \left(-\frac{1}{1}\right) \\ &= 1 - \frac{1}{t}\end{aligned}$$

Step 3: Take the limit

$$\lim_{t\to\infty}\left(1 - \frac{1}{t}\right) = 1 - 0 = 1$$

Answer: The integral **converges** to $\boxed{1}$.
Interpretation: The area under $y = \frac{1}{x^2}$ from $x = 1$ to ∞ is finite (equals 1)!

Example 54: Divergent Improper Integral

Problem: Determine whether $\int_1^\infty \frac{1}{x}\,dx$ converges or diverges.

Solution:
Step 1: Write as a limit

$$\int_1^\infty \frac{1}{x}\,dx = \lim_{t\to\infty} \int_1^t \frac{1}{x}\,dx$$

Step 2: Evaluate the proper integral

$$\begin{aligned}\int_1^t \frac{1}{x}\,dx &= [\ln|x|]_1^t \\ &= \ln t - \ln 1 \\ &= \ln t\end{aligned}$$

Step 3: Take the limit

$$\lim_{t\to\infty} \ln t = \infty$$

Answer: The integral **diverges**.

⚠ **Key Comparison:**

- $\int_1^\infty \frac{1}{x^2}\,dx$ converges
- $\int_1^\infty \frac{1}{x}\,dx$ diverges

The general rule: $\int_1^\infty \frac{1}{x^p}\,dx$ converges if and only if $p > 1$.

Theorem 5.19: p-Integral Test

For $p > 0$:

$$\int_1^\infty \frac{1}{x^p}\,dx = \begin{cases} \text{converges} & \text{if } p > 1 \\ \text{diverges} & \text{if } p \leq 1 \end{cases}$$

ℹ **Specific Values:**

For $p > 1$:

$$\int_1^\infty \frac{1}{x^p}\,dx = \frac{1}{p-1}$$

ℹ **Similar Result for Lower Limit:**

$$\int_0^1 \frac{1}{x^p}\,dx = \begin{cases} \text{converges} & \text{if } p < 1 \\ \text{diverges} & \text{if } p \geq 1 \end{cases}$$

This is the foundation for the integral test in series convergence.

Example 55: Improper Integral with Exponential

Problem: Evaluate $\displaystyle\int_0^\infty e^{-x}\,dx$.

Solution:

Step 1: Write as a limit

$$\int_0^\infty e^{-x}\,dx = \lim_{t\to\infty} \int_0^t e^{-x}\,dx$$

Step 2: Evaluate

$$\begin{aligned} \int_0^t e^{-x}\,dx &= [-e^{-x}]_0^t \\ &= -e^{-t} - (-e^0) \\ &= -e^{-t} + 1 \\ &= 1 - e^{-t} \end{aligned}$$

Step 3: Take the limit

$$\lim_{t\to\infty}(1-e^{-t}) = 1 - 0 = 1$$

Answer: $\boxed{1}$

Note: Exponential decay functions like e^{-x} typically give convergent improper integrals.

Theorem 5.20: Improper Integrals – Type 2 (Discontinuous Integrand)

If f is continuous on $[a, b)$ and discontinuous at b:

$$\boxed{\int_a^b f(x)\,dx = \lim_{t\to b^-}\int_a^t f(x)\,dx}$$

If f is continuous on $(a, b]$ and discontinuous at a:

$$\boxed{\int_a^b f(x)\,dx = \lim_{t\to a^+}\int_t^b f(x)\,dx}$$

If f has a discontinuity at c where $a < c < b$:

$$\boxed{\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx}$$

where both integrals on the right are improper and must converge.

⚠ **Watch For:**

Common discontinuities occur when the integrand has:

- Division by zero at an endpoint or interior point
- Logarithm of zero
- Vertical asymptotes

Example 56: Discontinuity at an Endpoint

Problem: Determine whether $\displaystyle\int_0^1 \frac{1}{\sqrt{x}}\,dx$ converges or diverges.

Solution:

Step 1: Identify the discontinuity

$f(x) = \frac{1}{\sqrt{x}}$ is discontinuous at $x = 0$ (the integrand approaches ∞).

Step 2: Write as a limit

$$\int_0^1 \frac{1}{\sqrt{x}}\,dx = \lim_{t\to 0^+}\int_t^1 \frac{1}{\sqrt{x}}\,dx$$

Step 3: Evaluate the proper integral

$$\int_t^1 \frac{1}{\sqrt{x}}\,dx = \int_t^1 x^{-1/2}\,dx$$
$$= \left[\frac{x^{1/2}}{1/2}\right]_t^1$$
$$= 2[x^{1/2}]_t^1$$
$$= 2[\sqrt{x}]_t^1$$
$$= 2(1-\sqrt{t})$$

Step 4: Take the limit

$$\lim_{t\to 0^+} 2(1-\sqrt{t}) = 2(1-0) = 2$$

Answer: The integral **converges** to $\boxed{2}$.

Pattern: This is the p-integral test at the lower limit with $p = 1/2 < 1$, so it converges.

Example 57: Discontinuity at Interior Point

Problem: Evaluate $\int_{-1}^{1} \frac{1}{x^2}\,dx$.

Solution:

Step 1: Identify the discontinuity

$f(x) = \frac{1}{x^2}$ has a discontinuity at $x = 0$ (inside the interval).

Step 2: Split at the discontinuity

$$\int_{-1}^{1} \frac{1}{x^2}\,dx = \int_{-1}^{0} \frac{1}{x^2}\,dx + \int_{0}^{1} \frac{1}{x^2}\,dx$$

Step 3: Evaluate each part

Left integral:

$$\int_{-1}^{0} \frac{1}{x^2}\,dx = \lim_{t\to 0^-} \int_{-1}^{t} \frac{1}{x^2}\,dx$$
$$= \lim_{t\to 0^-} \left[-\frac{1}{x}\right]_{-1}^{t}$$
$$= \lim_{t\to 0^-} \left(-\frac{1}{t}+1\right)$$
$$= -\infty + 1 = -\infty$$

Since the left integral diverges, we don't need to check the right integral.

Answer: The integral **diverges**.

Critical Error to Avoid:

Do NOT evaluate as:

$$\left[-\frac{1}{x}\right]_{-1}^{1} = -1 - 1 = -2 \quad \text{(WRONG!)}$$

You must recognize the discontinuity at $x = 0$ and split the integral!

Example 58: Comparison Test for Convergence

Problem: Determine whether $\int_1^\infty \frac{e^{-x}}{x}\,dx$ converges.

Solution:
Strategy: This integral is difficult to evaluate directly, so use comparison.
Step 1: Find a comparable function
For $x \geq 1$: $\frac{1}{x} \leq 1$, so:

$$\frac{e^{-x}}{x} \leq e^{-x}$$

Step 2: Check if the comparison integral converges
We know from Example 55 that:

$$\int_1^\infty e^{-x}\,dx = e^{-1} < \infty \quad \text{(converges)}$$

Step 3: Apply comparison test
Since $0 \leq \frac{e^{-x}}{x} \leq e^{-x}$ and $\int_1^\infty e^{-x}\,dx$ converges, by the comparison test:

$$\int_1^\infty \frac{e^{-x}}{x}\,dx \quad \text{converges}$$

Answer: The integral **converges** (though we don't have an exact value).
Comparison Test: If $0 \leq f(x) \leq g(x)$ and $\int g(x)\,dx$ converges, then $\int f(x)\,dx$ converges.

Section 5.13 Summary: Improper Integrals

Type 1 – Infinite Limits:

$$\boxed{\int_a^\infty f(x)\,dx = \lim_{t\to\infty}\int_a^t f(x)\,dx}$$

Type 2 – Discontinuous Integrand:

$$\boxed{\int_a^b f(x)\,dx = \lim_{t\to c}\int_a^t f(x)\,dx}$$

where f is discontinuous at c.
p-Integral Tests:

$$\int_1^\infty \frac{1}{x^p}\,dx : \quad \text{converges if } p > 1, \text{ diverges if } p \leq 1$$
$$\int_0^1 \frac{1}{x^p}\,dx : \quad \text{converges if } p < 1, \text{ diverges if } p \geq 1$$

Procedure:

1. Identify why the integral is improper (infinite limit or discontinuity)
2. Write as appropriate limit

3. Evaluate the proper integral
4. Take the limit
5. Determine convergence or divergence

⚠ **Common Mistakes:**

- Forgetting to check for discontinuities in the integrand
- Not splitting at interior discontinuities
- Evaluating $\int_{-\infty}^{\infty}$ as $\lim_{t\to\infty}\int_{-t}^{t}$ (must split!)
- Forgetting the limit notation entirely

Quick Tests:

- $\int_1^\infty e^{-x}\,dx$ converges
- $\int_1^\infty \frac{1}{x}\,dx$ diverges
- $\int_1^\infty \frac{1}{x^2}\,dx$ converges

Practice Exercises

Problem 1: Indefinite Integrals – Basic Rules

Evaluate the following indefinite integrals:

(a) $\displaystyle\int (4x^3 - 6x^2 + 2x - 5)\,dx$

(b) $\displaystyle\int \left(3\sqrt{x} - \frac{2}{x^3}\right) dx$

(c) $\displaystyle\int \frac{x^4 + 3x^2 - 1}{x^2}\,dx$

(d) $\displaystyle\int (e^x + 3^x - \sin x)\,dx$

(e) $\displaystyle\int \left(\sec^2 x + \frac{1}{x}\right) dx$

(f) $\displaystyle\int \frac{5}{1 + x^2}\,dx$

Problem 2: Initial Value Problems

Find the function f satisfying the given conditions:

(a) $f'(x) = 3x^2 - 4x + 1$ and $f(1) = 2$

(b) $f'(x) = \sin x + e^x$ and $f(0) = 5$

(c) $f''(x) = 12x^2 - 6x$, $f'(0) = 2$, and $f(0) = 1$

(d) $f'(x) = \dfrac{2x + 1}{\sqrt{x}}$ and $f(4) = 10$

(e) A particle has acceleration $a(t) = 6t - 4$ m/s². At $t = 0$, velocity is 3 m/s and position is 5 m. Find position as a function of time.

Problem 3: Mixed Basic Integrals

Evaluate:

(a) $\displaystyle\int_0^2 (x^2 + 2x)\,dx$

(b) $\displaystyle\int_1^4 \sqrt{x}\,dx$

(c) $\displaystyle\int_{-\pi/4}^{\pi/4} \cos x\,dx$

(d) $\displaystyle\int_0^1 (e^x + x^e)\,dx$

(e) $\displaystyle\int_1^e \frac{1}{x}\,dx$

Problem 4: Basic Substitution

Evaluate using substitution:

(a) $\displaystyle\int (3x + 1)^{10}\,dx$

(b) $\displaystyle\int x^2(x^3 + 5)^7\,dx$

(c) $\displaystyle\int \frac{x}{\sqrt{x^2+9}}\,dx$

(d) $\displaystyle\int \sin^4 x\cos x\,dx$

(e) $\displaystyle\int xe^{x^2}\,dx$

(f) $\displaystyle\int \frac{\ln x}{x}\,dx$

Problem 5: Substitution with Definite Integrals

Evaluate by substitution (change limits or substitute back):

(a) $\displaystyle\int_0^2 x(x^2+1)^3\,dx$

(b) $\displaystyle\int_0^{\pi/2} \sin x\cos^3 x\,dx$

(c) $\displaystyle\int_1^e \frac{(\ln x)^2}{x}\,dx$

(d) $\displaystyle\int_0^1 \frac{2x}{(x^2+1)^2}\,dx$

(e) $\displaystyle\int_0^{\pi/4} \tan x\sec^2 x\,dx$

Problem 6: Advanced Substitution

Evaluate:

(a) $\displaystyle\int \frac{e^{\sqrt{x}}}{\sqrt{x}}\,dx$

(b) $\displaystyle\int \frac{x^3}{\sqrt{x^2+1}}\,dx$

(c) $\displaystyle\int \frac{x}{(x+1)^2}\,dx$

(d) $\displaystyle\int_0^{\ln 2} \frac{e^x}{1+e^x}\,dx$

(e) $\displaystyle\int \frac{\cos(\ln x)}{x}\,dx$

Problem 7: Substitution with Trigonometry

Evaluate:

(a) $\displaystyle\int \frac{\sin x}{1+\cos^2 x}\,dx$

(b) $\displaystyle\int e^x\sin(e^x)\,dx$

(c) $\displaystyle\int \frac{\sec^2 x}{\tan x}\,dx$

(d) $\displaystyle\int_0^1 \frac{x}{\sqrt{1-x^4}}\,dx$

Problem 8: Basic Integration by Parts

Evaluate using integration by parts:

(a) $\displaystyle\int xe^{2x}\,dx$

(b) $\displaystyle\int x\cos x\,dx$

(c) $\displaystyle\int \ln(2x)\,dx$

(d) $\displaystyle\int x^2\sin x\,dx$

(e) $\displaystyle\int e^x\cos x\,dx$

(f) $\displaystyle\int \arctan x\,dx$

Problem 9: Integration by Parts – Definite Integrals

Evaluate:

(a) $\displaystyle\int_0^1 xe^x\,dx$

(b) $\displaystyle\int_1^e x\ln x\,dx$

(c) $\displaystyle\int_0^{\pi/2} x\sin x\,dx$

(d) $\displaystyle\int_0^1 x^3e^{-x}\,dx$

Problem 10: Repeated Integration by Parts

Evaluate:

(a) $\displaystyle\int x^3e^x\,dx$

(b) $\displaystyle\int x^2\cos x\,dx$

(c) $\displaystyle\int e^{2x}\sin(3x)\,dx$

(d) $\displaystyle\int_0^{\pi} x^2\sin x\,dx$

Problem 11: Powers of Sine and Cosine

Evaluate:

(a) $\displaystyle\int \sin^5 x\,dx$

(b) $\displaystyle\int \cos^4 x\,dx$

(c) $\displaystyle\int \sin^3 x\cos^4 x\,dx$

(d) $\displaystyle\int_0^{\pi/2} \sin^2 x\cos^2 x\,dx$

(e) $\int \sin^2(3x)\, dx$

Problem 12: Tangent and Secant

Evaluate:

(a) $\int \tan^2 x\, dx$

(b) $\int \tan^3 x \sec^2 x\, dx$

(c) $\int \sec^3 x \tan x\, dx$

(d) $\int_0^{\pi/4} \sec^4 x\, dx$

(e) $\int \tan x\, dx$

Problem 13: Mixed Trigonometric Integrals

Evaluate:

(a) $\int \sin(2x)\cos(4x)\, dx$

(b) $\int \sin^2 x \cos^3 x\, dx$

(c) $\int_0^{\pi} \sin^4 x\, dx$

(d) $\int \frac{1}{\sin x \cos x}\, dx$

Problem 14: Partial Fractions – Distinct Linear

Evaluate using partial fractions:

(a) $\int \frac{3x-1}{(x-1)(x+2)}\, dx$

(b) $\int \frac{x+7}{x^2+x-6}\, dx$

(c) $\int_2^4 \frac{1}{x^2-1}\, dx$

(d) $\int \frac{5x+7}{2x^2+7x+3}\, dx$

Problem 15: Partial Fractions – Repeated/Irreducible

Evaluate:

(a) $\int \frac{x}{(x-1)^2(x+1)}\, dx$

(b) $\int \frac{x^2+1}{(x-1)(x^2+1)}\, dx$

(c) $\int \frac{3x^2-2x+1}{x^3+x}\, dx$

(d) $\displaystyle\int \frac{2x+3}{(x+1)^3}\,dx$

Problem 16: Partial Fractions – Improper

Evaluate (use long division first if needed):

(a) $\displaystyle\int \frac{x^3+1}{x^2-1}\,dx$

(b) $\displaystyle\int \frac{x^3}{x^2+x-2}\,dx$

(c) $\displaystyle\int_2^3 \frac{x^2}{x-1}\,dx$

Problem 17: Riemann Sums

(a) Approximate $\displaystyle\int_0^4 x^2\,dx$ using a left Riemann sum with $n = 4$ rectangles.

(b) Approximate $\displaystyle\int_1^3 \frac{1}{x}\,dx$ using a right Riemann sum with $n = 4$.

(c) Approximate $\displaystyle\int_0^2 e^x\,dx$ using a midpoint Riemann sum with $n = 4$.

(d) For $f(x) = x^2$ on $[0, 3]$, find the exact value using a limit of Riemann sums.

Problem 18: Properties of Definite Integrals

Given $\displaystyle\int_0^3 f(x)\,dx = 7$ and $\displaystyle\int_0^3 g(x)\,dx = -2$:

(a) Find $\displaystyle\int_0^3 [2f(x) - 3g(x)]\,dx$

(b) Find $\displaystyle\int_3^0 f(x)\,dx$

(c) If $\displaystyle\int_0^1 f(x)\,dx = 3$, find $\displaystyle\int_1^3 f(x)\,dx$

(d) If $f(x) \geq 0$ and $g(x) \leq 0$ on $[0, 3]$, what can you say about $\int_0^3 [f(x) - g(x)]\,dx$?

Problem 19: FTC Part 1

Find the derivative:

(a) $\displaystyle\frac{d}{dx}\left[\int_2^x t^3\,dt\right]$

(b) $\displaystyle\frac{d}{dx}\left[\int_0^x \sin(t^2)\,dt\right]$

(c) $\displaystyle\frac{d}{dx}\left[\int_0^{x^2} \sqrt{1+t^3}\,dt\right]$

(d) $\displaystyle\frac{d}{dx}\left[\int_x^1 e^{-t^2}\,dt\right]$

(e) $\dfrac{d}{dx}\left[\displaystyle\int_{\sin x}^{\cos x} t^2\,dt\right]$

Problem 20: FTC Part 2 – Basic

Evaluate using the Fundamental Theorem:

(a) $\displaystyle\int_1^4 \frac{1}{\sqrt{x}}\,dx$

(b) $\displaystyle\int_{-1}^1 (x^3 - 3x)\,dx$

(c) $\displaystyle\int_0^\pi (2\sin x + \cos x)\,dx$

(d) $\displaystyle\int_1^2 \left(x^2 + \frac{1}{x^2}\right)dx$

(e) $\displaystyle\int_0^{\ln 2} e^{2x}\,dx$

Problem 21: Net Change Problems

(a) A particle moves with velocity $v(t) = 3t^2 - 12t + 9$ m/s. Find:

- Displacement from $t = 0$ to $t = 4$
- Total distance traveled from $t = 0$ to $t = 4$

(b) Water flows into a tank at rate $r(t) = 100 - 2t$ liters/minute. How much water enters from $t = 0$ to $t = 30$ minutes?

(c) A bacteria population grows at rate $P'(t) = 500e^{0.1t}$ bacteria/hour. Find the increase in population from $t = 0$ to $t = 10$ hours.

Problem 22: Average Value

(a) Find the average value of $f(x) = x^2$ on $[0, 3]$.

(b) Find the average value of $f(x) = \sin x$ on $[0, \pi]$.

(c) At what point(s) does $f(x) = x^2$ on $[1, 3]$ equal its average value?

(d) Find c such that the average value of $f(x) = x$ on $[0, c]$ equals 2.

Problem 23: Area Under Curves

Find the area of the region:

(a) Bounded by $y = x^2 + 1$, the x-axis, $x = 0$, and $x = 2$

(b) Bounded by $y = \sin x$, the x-axis, from $x = 0$ to $x = \pi$

(c) Bounded by $y = e^x$, the x-axis, $x = 0$, and $x = 2$

(d) Bounded by $y = \frac{1}{x}$, the x-axis, $x = 1$, and $x = e$

Problem 24: Area Between Two Curves – Basic

Find the area between:

(a) $y = x^2$ and $y = x + 2$
(b) $y = x^2 - 4$ and $y = 0$ (the x-axis)
(c) $y = \sqrt{x}$ and $y = x^2$
(d) $y = x^3$ and $y = x$ (from $x = -1$ to $x = 1$)
(e) $y = e^x$ and $y = 1$ from $x = 0$ to $x = 1$

Problem 25: Area – More Complex

Find the enclosed area:

(a) Between $y = 2x$ and $y = x^2 - 4x$
(b) Between $y = \sin x$ and $y = \cos x$ from $x = 0$ to $x = \pi/4$
(c) Between $y = |x|$ and $y = x^2 - 2$
(d) Completely enclosed by $y = x^2$ and $y = 2x - x^2$
(e) Between $y = \ln x$, the x-axis, $x = 1$, and $x = e$

Problem 26: Area with Mixed Regions

(a) Find the area bounded by $y = x^3 - x$ and the x-axis.
Hint: The curve crosses the x-axis at $x = -1, 0, 1$. Consider regions separately.
(b) Find the total area between $y = \cos x$ and the x-axis from $x = 0$ to $x = 2\pi$.
(c) Sketch and find the area of the region bounded by $y = x^2 - 2x$ and $y = x$.

Problem 27: Disk Method

Find the volume when the region is rotated about the x-axis:

(a) Bounded by $y = x^2$, $y = 0$, $x = 0$, and $x = 2$
(b) Bounded by $y = \sqrt{x}$, $y = 0$, $x = 0$, and $x = 4$
(c) Bounded by $y = e^x$, $y = 0$, $x = 0$, and $x = 1$
(d) Bounded by $y = \sin x$, $y = 0$, from $x = 0$ to $x = \pi$

Problem 28: Washer Method

Find the volume when rotated about the x-axis:

(a) Region between $y = x$ and $y = x^2$
(b) Region between $y = \sqrt{x}$ and $y = x^2$ from $x = 0$ to $x = 1$
(c) Region between $y = 2$ and $y = x^2$ from $x = -1$ to $x = 1$
(d) Region between $y = \sin x$ and $y = \cos x$ from $x = 0$ to $x = \pi/4$

Problem 29: Shell Method

Find the volume when rotated about the y-axis:

(a) Region bounded by $y = x^2$, $y = 0$, and $x = 2$
(b) Region between $y = x$ and $y = x^2$
(c) Region bounded by $y = \sqrt{x}$, $y = 0$, and $x = 4$
(d) Region between $y = x^2$ and $y = 4$ from $x = 0$ to $x = 2$

Problem 30: Mixed Volume Problems

(a) A solid has circular cross-sections perpendicular to the x-axis. The radius at position x is $r(x) = \sqrt{x}$ for $0 \leq x \leq 4$. Find the volume.

(b) Find the volume of a sphere of radius r by rotating $y = \sqrt{r^2 - x^2}$ about the x-axis.

(c) The region bounded by $y = x^2$ and $y = 4$ is rotated about the line $y = 5$. Find the volume.

(d) Compare the volumes when the region between $y = x$ and $y = x^2$ is rotated about:

- The x-axis
- The y-axis
- The line $x = 2$

Problem 31: Arc Length

Find the length of the curve:

(a) $y = \frac{2}{3}x^{3/2}$ from $x = 0$ to $x = 3$

(b) $y = \frac{x^3}{3} + \frac{1}{4x}$ from $x = 1$ to $x = 3$

(c) $y = \ln(\cos x)$ from $x = 0$ to $x = \pi/4$

(d) $y = x^2$ from $x = 0$ to $x = 1$ (set up the integral; exact evaluation difficult)

Problem 32: Surface Area

Find the surface area when rotated about the x-axis:

(a) $y = x^3$ from $x = 0$ to $x = 1$

(b) $y = \sqrt{x}$ from $x = 1$ to $x = 4$

(c) $y = x^2$ from $x = 0$ to $x = 2$ (set up the integral)

(d) Verify the surface area formula for a cylinder: rotate $y = r$ from $x = 0$ to $x = h$.

Problem 33: Infinite Limits

Determine convergence or divergence. If convergent, find the value:

(a) $\displaystyle\int_1^\infty \frac{1}{x^3}\,dx$

(b) $\displaystyle\int_1^\infty \frac{1}{\sqrt{x}}\,dx$

(c) $\displaystyle\int_0^\infty e^{-2x}\,dx$

(d) $\displaystyle\int_2^\infty \frac{1}{x(\ln x)^2}\,dx$

(e) $\displaystyle\int_{-\infty}^\infty \frac{1}{1+x^2}\,dx$

(f) $\displaystyle\int_1^\infty \frac{\ln x}{x^2}\,dx$

Problem 34: Discontinuous Integrands

Determine convergence or divergence:

(a) $\displaystyle\int_0^1 \frac{1}{\sqrt[3]{x}}\,dx$

(b) $\displaystyle\int_0^1 \frac{1}{x^2}\,dx$

(c) $\displaystyle\int_{-1}^1 \frac{1}{x^{2/3}}\,dx$

(d) $\displaystyle\int_0^{\pi/2} \sec x\,dx$

(e) $\displaystyle\int_0^1 \ln x\,dx$

(f) $\displaystyle\int_{-1}^3 \frac{1}{(x-1)^2}\,dx$

Problem 35: Mixed Improper Integrals

(a) For what values of p does $\displaystyle\int_1^\infty \frac{1}{x^p}\,dx$ converge?

(b) For what values of p does $\displaystyle\int_0^1 \frac{1}{x^p}\,dx$ converge?

(c) Determine convergence: $\displaystyle\int_0^\infty xe^{-x^2}\,dx$

(d) Determine convergence: $\displaystyle\int_1^\infty \frac{\sin^2 x}{x^2}\,dx$ (use comparison test)

(e) Does $\displaystyle\int_0^\infty \sin x\,dx$ converge? Explain.

Chapter 5 Practice Problems Summary

Coverage by Topic:

Topic	**Problems**	**Exam Frequency**
Substitution	4–7	**Every Exam**
Area Between Curves	23–26	**Very Frequent**
FTC & Evaluation	19–22	**Every Exam**
Volumes of Revolution	27–30	**Very Frequent**
Integration by Parts	8–10	High
Trigonometric Integrals	11–13	High
Partial Fractions	14–16	Medium-High
Improper Integrals	33–35	Medium
Arc Length/Surface Area	31–32	Medium
Riemann Sums	17–18	Low-Medium

Chapter 6

Vectors and Analytic Geometry in $\mathbb{R}^3$

Vectors in 2D and 3D

Vectors are fundamental objects in mathematics and physics. Unlike scalars (which have only magnitude), vectors have both magnitude and direction. This section introduces vectors in two and three dimensions.

Definition 6.1: Vectors

A **vector** is a quantity that has both magnitude (length) and direction. We represent vectors geometrically by directed line segments (arrows).

Component Form in 2D:

$$\vec{v} = \langle v_1, v_2 \rangle \quad \text{or} \quad \vec{v} = v_1\vec{i} + v_2\vec{j}$$

where $\vec{i} = \langle 1, 0 \rangle$ and $\vec{j} = \langle 0, 1 \rangle$ are the standard unit vectors.

Component Form in 3D:

$$\vec{v} = \langle v_1, v_2, v_3 \rangle \quad \text{or} \quad \vec{v} = v_1\vec{i} + v_2\vec{j} + v_3\vec{k}$$

where $\vec{i} = \langle 1, 0, 0 \rangle$, $\vec{j} = \langle 0, 1, 0 \rangle$, $\vec{k} = \langle 0, 0, 1 \rangle$ are standard unit vectors.

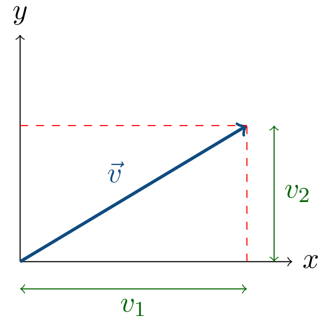
Notation:

- Boldface: $\mathbf{v}$ or arrow: $\vec{v}$ or $\overrightarrow{v}$
- Components: v_1, v_2, v_3 (or v_x, v_y, v_z)

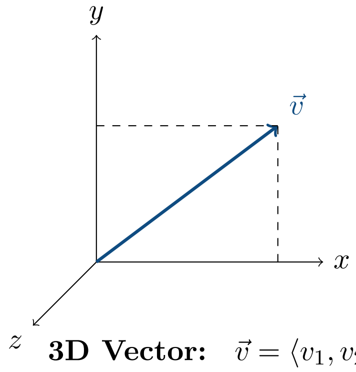
Position Vector:

The vector from the origin O to point $P(x, y, z)$ is:

$$\overrightarrow{OP} = \langle x, y, z \rangle$$



2D Vector: $\vec{v} = \langle v_1, v_2 \rangle$



3D Vector: $\vec{v} = \langle v_1, v_2, v_3 \rangle$

Definition 6.2: Magnitude (Length) of a Vector

The **magnitude** (or **length** or **norm**) of a vector $\vec{v}$ is denoted $|\vec{v}|$ or $\|\vec{v}\|$.

In 2D:

$$\boxed{|\vec{v}| = \sqrt{v_1^2 + v_2^2}}$$

In 3D:

$$\boxed{|\vec{v}| = \sqrt{v_1^2 + v_2^2 + v_3^2}}$$

Properties:

- $|\vec{v}| \geq 0$ (always non-negative)
- $|\vec{v}| = 0$ if and only if $\vec{v} = \vec{0}$ (zero vector)
- $|\vec{v}|$ represents the distance from the origin to the point (v_1, v_2, v_3)

Distance Between Two Points:

The distance between points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ is:

$$d(P, Q) = |\overrightarrow{PQ}| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

Example 1: Finding Magnitude

Problem: Find the magnitude of each vector:

(a) $\vec{v} = \langle 3, -4 \rangle$
(b) $\vec{w} = \langle 1, 2, -2 \rangle$
(c) $\vec{u} = 2\vec{i} - 3\vec{j} + 6\vec{k}$

Solution:

(a) Using the 2D magnitude formula:

$$|\vec{v}| = \sqrt{3^2 + (-4)^2} = \sqrt{9 + 16} = \sqrt{25} = 5$$

Answer (a): $\boxed{|\vec{v}| = 5}$

(b) Using the 3D magnitude formula:

$$|\vec{w}| = \sqrt{1^2 + 2^2 + (-2)^2} = \sqrt{1 + 4 + 4} = \sqrt{9} = 3$$

Answer (b): $\boxed{|\vec{w}| = 3}$

(c) First write in component form: $\vec{u} = \langle 2, -3, 6 \rangle$

$$|\vec{u}| = \sqrt{2^2 + (-3)^2 + 6^2} = \sqrt{4 + 9 + 36} = \sqrt{49} = 7$$

Answer (c): $\boxed{|\vec{u}| = 7}$

Definition 6.3: Direction of a Vector

The **direction** of a vector $\vec{v}$ is specified by:

1. Unit Vector:
A **unit vector** in the direction of $\vec{v}$ is:

$$\boxed{\hat{v} = \frac{\vec{v}}{|\vec{v}|} = \frac{1}{|\vec{v}|}\vec{v}}$$

This has magnitude 1 and points in the same direction as $\vec{v}$.

2. Direction Angles (in 3D):
The angles α, β, γ that $\vec{v}$ makes with the positive x-, y-, and z-axes:

$$\cos\alpha = \frac{v_1}{|\vec{v}|}, \quad \cos\beta = \frac{v_2}{|\vec{v}|}, \quad \cos\gamma = \frac{v_3}{|\vec{v}|}$$

These are called **direction cosines**.

3. Direction Angles Property:

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$$

Note:
The unit vector $\hat{v} = \langle \cos\alpha, \cos\beta, \cos\gamma \rangle$

Example 2: Finding Unit Vectors

Problem: Find the unit vector in the direction of $\vec{v} = \langle 2, -1, 2 \rangle$.

Solution:
Step 1: Find the magnitude

$$|\vec{v}| = \sqrt{2^2 + (-1)^2 + 2^2} = \sqrt{4 + 1 + 4} = \sqrt{9} = 3$$

Step 2: Divide by magnitude

$$\hat{v} = \frac{\vec{v}}{|\vec{v}|} = \frac{1}{3}\langle 2, -1, 2 \rangle = \left\langle \frac{2}{3}, -\frac{1}{3}, \frac{2}{3} \right\rangle$$

Answer: $\boxed{\hat{v} = \left\langle \frac{2}{3}, -\frac{1}{3}, \frac{2}{3} \right\rangle}$

Verification: $|\hat{v}| = \sqrt{\frac{4}{9} + \frac{1}{9} + \frac{4}{9}} = \sqrt{\frac{9}{9}} = 1$ ✓

Example 3: Vector Between Two Points

Problem: Find the vector from $P(1,2,3)$ to $Q(4,-1,5)$ and its magnitude.

Solution:
Step 1: Find the vector

$$\overrightarrow{PQ} = \langle 4-1, -1-2, 5-3 \rangle = \langle 3, -3, 2 \rangle$$

Step 2: Find magnitude

$$|\overrightarrow{PQ}| = \sqrt{3^2 + (-3)^2 + 2^2} = \sqrt{9+9+4} = \sqrt{22}$$

Answer:

- Vector: $\boxed{\overrightarrow{PQ} = \langle 3, -3, 2 \rangle}$
- Distance: $\boxed{d(P,Q) = \sqrt{22}}$

Note: The magnitude $|\overrightarrow{PQ}|$ is the distance between P and Q.

Definition 6.4: Special Vectors

Zero Vector:

$$\vec{0} = \langle 0, 0, 0 \rangle$$

Has magnitude 0 and no specific direction.
Standard Unit Vectors:

$$\begin{aligned} \vec{i} &= \langle 1, 0, 0 \rangle \quad \text{(along positive } x\text{-axis)} \\ \vec{j} &= \langle 0, 1, 0 \rangle \quad \text{(along positive } y\text{-axis)} \\ \vec{k} &= \langle 0, 0, 1 \rangle \quad \text{(along positive } z\text{-axis)} \end{aligned}$$

Any vector can be written as:

$$\vec{v} = v_1\vec{i} + v_2\vec{j} + v_3\vec{k}$$

Parallel Vectors:
Two non-zero vectors $\vec{u}$ and $\vec{v}$ are **parallel** if:

$$\vec{v} = c\vec{u} \quad \text{for some scalar } c \neq 0$$

If $c > 0$: same direction; if $c < 0$: opposite directions.

Section 6.1 Summary: Vectors in 2D and 3D
Component Form:

$$\begin{aligned} &\text{2D:} \quad \vec{v} = \langle v_1, v_2 \rangle = v_1\vec{i} + v_2\vec{j} \\ &\text{3D:} \quad \vec{v} = \langle v_1, v_2, v_3 \rangle = v_1\vec{i} + v_2\vec{j} + v_3\vec{k} \end{aligned}$$

Magnitude:

$$\boxed{|\vec{v}| = \sqrt{v_1^2 + v_2^2 + v_3^2}}$$

Unit Vector:

$$\boxed{\hat{v} = \frac{\vec{v}}{|\vec{v}|}}$$

Vector Between Points:

$$\overrightarrow{PQ} = \langle x_2 - x_1, y_2 - y_1, z_2 - z_1 \rangle$$

Distance Formula:

$$d(P, Q) = |\overrightarrow{PQ}| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

⚠ **Common Mistakes:**

- Forgetting to take square root when finding magnitude
- Sign errors when finding vector between two points (order matters!)
- Not checking that unit vectors have magnitude 1

Quick Check: A unit vector always has magnitude equal to 1.

Vector Operations

Vectors can be added together and multiplied by scalars. These operations have both algebraic and geometric interpretations.

Definition 6.5: Vector Addition

Given $\vec{u} = \langle u_1, u_2, u_3 \rangle$ and $\vec{v} = \langle v_1, v_2, v_3 \rangle$:

Addition:

$$\boxed{\vec{u} + \vec{v} = \langle u_1 + v_1, u_2 + v_2, u_3 + v_3 \rangle}$$

Add corresponding components.

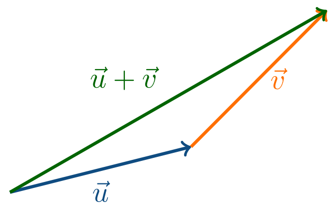
Geometric Interpretation:

1. Triangle Method: Place tail of $\vec{v}$ at head of $\vec{u}$. The sum $\vec{u} + \vec{v}$ goes from tail of $\vec{u}$ to head of $\vec{v}$.

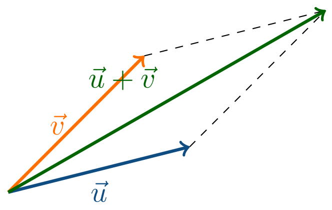
2. Parallelogram Method: Place both vectors at the same initial point. The sum is the diagonal of the parallelogram formed.

Properties:

- **Commutative:** $\vec{u} + \vec{v} = \vec{v} + \vec{u}$
- **Associative:** $(\vec{u} + \vec{v}) + \vec{w} = \vec{u} + (\vec{v} + \vec{w})$
- **Identity:** $\vec{v} + \vec{0} = \vec{v}$
- **Inverse:** $\vec{v} + (-\vec{v}) = \vec{0}$



Triangle Method



Parallelogram Method

Definition 6.6: Scalar Multiplication

Given a vector $\vec{v} = \langle v_1, v_2, v_3 \rangle$ and scalar c:

Scalar Multiplication:

$$\boxed{c\vec{v} = \langle cv_1, cv_2, cv_3 \rangle}$$

Multiply each component by c.

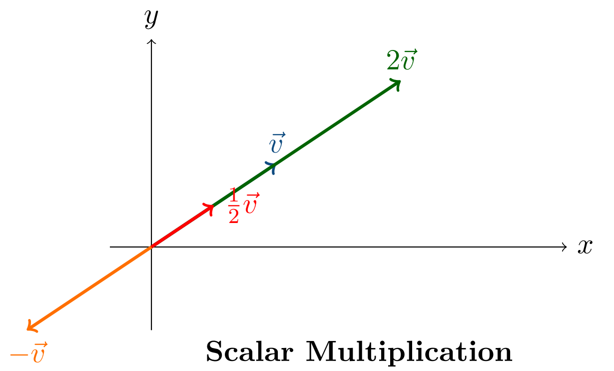
Geometric Interpretation:

- If $c > 0$: $c\vec{v}$ has the same direction as $\vec{v}$, but length is scaled by $|c|$
- If $c < 0$: $c\vec{v}$ has the opposite direction, length scaled by $|c|$
- If $c = 0$: $c\vec{v} = \vec{0}$

Properties:

- $c(\vec{u} + \vec{v}) = c\vec{u} + c\vec{v}$ (distributive)

- $(c+d)\vec{v} = c\vec{v} + d\vec{v}$ (distributive)
- $(cd)\vec{v} = c(d\vec{v})$ (associative)
- $1\vec{v} = \vec{v}$
- $|c\vec{v}| = |c||\vec{v}|$



Scalar Multiplication
Different scalars change length and direction

Example 4: Vector Addition and Scalar Multiplication

Problem: Given $\vec{u} = \langle 2, -1, 3\rangle$ and $\vec{v} = \langle -1, 4, 2\rangle$, find:
(a) $\vec{u} + \vec{v}$
(b) $\vec{u} - \vec{v}$
(c) $3\vec{u}$
(d) $2\vec{u} - 3\vec{v}$

Solution:
(a) Add corresponding components:

$$\vec{u} + \vec{v} = \langle 2 + (-1), -1 + 4, 3 + 2\rangle = \langle 1, 3, 5\rangle$$

Answer (a): $\boxed{\langle 1, 3, 5\rangle}$

(b) Subtract corresponding components:

$$\vec{u} - \vec{v} = \langle 2 - (-1), -1 - 4, 3 - 2\rangle = \langle 3, -5, 1\rangle$$

Answer (b): $\boxed{\langle 3, -5, 1\rangle}$

(c) Multiply each component by 3:

$$3\vec{u} = \langle 3(2), 3(-1), 3(3)\rangle = \langle 6, -3, 9\rangle$$

Answer (c): $\boxed{\langle 6, -3, 9\rangle}$

(d) Combine operations:

$$\begin{aligned} 2\vec{u} - 3\vec{v} &= 2\langle 2, -1, 3\rangle - 3\langle -1, 4, 2\rangle \\ &= \langle 4, -2, 6\rangle - \langle -3, 12, 6\rangle \\ &= \langle 4 - (-3), -2 - 12, 6 - 6\rangle \\ &= \langle 7, -14, 0\rangle \end{aligned}$$

Answer (d): $\boxed{\langle 7, -14, 0\rangle}$

Definition 6.7: Vector Subtraction

Subtraction:

$$\boxed{\vec{u} - \vec{v} = \vec{u} + (-\vec{v}) = \langle u_1 - v_1, u_2 - v_2, u_3 - v_3\rangle}$$

Geometric Interpretation:
$\vec{u} - \vec{v}$ is the vector from the head of $\vec{v}$ to the head of $\vec{u}$ (when both start at the same point).

Key Formula:
The vector from point A to point B is:

$$\overrightarrow{AB} = \vec{b} - \vec{a}$$

where $\vec{a}$ and $\vec{b}$ are position vectors of A and B.

Example 5: Finding a Parallel Vector

Problem: Find a vector of length 10 that is parallel to $\vec{v} = \langle 3, -4, 0\rangle$.

Solution:
Step 1: Find the unit vector in the direction of $\vec{v}$

$$|\vec{v}| = \sqrt{3^2 + (-4)^2 + 0^2} = \sqrt{9 + 16} = 5$$
$$\hat{v} = \frac{\vec{v}}{|\vec{v}|} = \frac{1}{5}\langle 3, -4, 0\rangle = \left\langle \frac{3}{5}, -\frac{4}{5}, 0 \right\rangle$$

Step 2: Scale to desired length

$$\vec{w} = 10\hat{v} = 10\left\langle \frac{3}{5}, -\frac{4}{5}, 0 \right\rangle = \langle 6, -8, 0\rangle$$

Verification: $|\vec{w}| = \sqrt{36 + 64 + 0} = \sqrt{100} = 10$ ✓
Answer: $\boxed{\vec{w} = \langle 6, -8, 0\rangle}$
Note: Another valid answer is $-\vec{w} = \langle -6, 8, 0\rangle$ (opposite direction, same length).

Example 6: Linear Combinations

Problem: Express $\vec{w} = \langle 7, -1, 10\rangle$ as a linear combination of $\vec{u} = \langle 1, 2, 1\rangle$ and $\vec{v} = \langle 2, -1, 3\rangle$.
That is, find a and b such that $\vec{w} = a\vec{u} + b\vec{v}$.

Solution:
Step 1: Set up the equation

$$\langle 7, -1, 10\rangle = a\langle 1, 2, 1\rangle + b\langle 2, -1, 3\rangle$$

Step 2: Write component equations

$$\begin{aligned} a + 2b &= 7 \quad (x\text{-component}) \\ 2a - b &= -1 \quad (y\text{-component}) \\ a + 3b &= 10 \quad (z\text{-component}) \end{aligned}$$

Step 3: Solve the system
From equation 1: $a = 7 - 2b$
Substitute into equation 2:

$$\begin{aligned} 2(7 - 2b) - b &= -1 \\ 14 - 4b - b &= -1 \\ -5b &= -15 \\ b &= 3 \end{aligned}$$

Then: $a = 7 - 2(3) = 1$
Step 4: Verify with third equation

$$a + 3b = 1 + 3(3) = 1 + 9 = 10 \quad \checkmark$$

Answer: $\boxed{\vec{w} = \vec{u} + 3\vec{v}}$
That is, $a = 1$ and $b = 3$.

Theorem 6.1: Properties of Vector Operations

Let $\vec{u}$, $\vec{v}$, $\vec{w}$ be vectors and c, d be scalars.

Addition Properties:

1. $\vec{u} + \vec{v} = \vec{v} + \vec{u}$ (commutative)
2. $(\vec{u} + \vec{v}) + \vec{w} = \vec{u} + (\vec{v} + \vec{w})$ (associative)
3. $\vec{u} + \vec{0} = \vec{u}$ (identity)
4. $\vec{u} + (-\vec{u}) = \vec{0}$ (inverse)

Scalar Multiplication Properties:

5. $c(\vec{u} + \vec{v}) = c\vec{u} + c\vec{v}$ (distributive)
6. $(c + d)\vec{u} = c\vec{u} + d\vec{u}$ (distributive)
7. $c(d\vec{u}) = (cd)\vec{u}$ (associative)
8. $1\vec{u} = \vec{u}$ (identity)

Magnitude Properties:

9. $|c\vec{u}| = |c||\vec{u}|$
10. $|\vec{u} + \vec{v}| \leq |\vec{u}| + |\vec{v}|$ (triangle inequality)

Section 6.2 Summary: Vector Operations

Vector Addition:

$$\boxed{\vec{u} + \vec{v} = \langle u_1 + v_1, u_2 + v_2, u_3 + v_3 \rangle}$$

Scalar Multiplication:

$$\boxed{c\vec{v} = \langle cv_1, cv_2, cv_3 \rangle}$$

Vector Subtraction:

$$\boxed{\vec{u} - \vec{v} = \langle u_1 - v_1, u_2 - v_2, u_3 - v_3 \rangle}$$

Key Properties:

- Addition is commutative and associative
- $|c\vec{v}| = |c||\vec{v}|$
- To find vector of specific length in direction of $\vec{v}$: $L\left(\frac{\vec{v}}{|\vec{v}|}\right)$

Geometric Interpretations:

- Addition: Triangle or parallelogram method
- Scalar multiplication: Stretching/shrinking and possibly reversing direction
- Subtraction: Vector from head of second to head of first

Common Mistakes:

- Confusing $\vec{u} - \vec{v}$ with $\vec{v} - \vec{u}$ (order matters!)
- Not distributing scalar multiplication to all components
- Forgetting absolute value when computing $|c\vec{v}| = |c||\vec{v}|$

Quick Check:

- $\vec{u} + \vec{v} = \vec{v} + \vec{u}$ (commutative)
- $2\vec{v}$ has same direction as $\vec{v}$ but twice the length
- $-\vec{v}$ has opposite direction and same length as $\vec{v}$

Dot Product

The dot product (also called scalar product or inner product) is a fundamental operation that takes two vectors and produces a scalar. It has important applications in computing angles, projections, and work.

Definition 6.8: Dot Product

Given $\vec{u} = \langle u_1, u_2, u_3 \rangle$ and $\vec{v} = \langle v_1, v_2, v_3 \rangle$:

Algebraic Definition:

$$\boxed{\vec{u} \cdot \vec{v} = u_1 v_1 + u_2 v_2 + u_3 v_3}$$

Multiply corresponding components and add.

Geometric Definition:

$$\boxed{\vec{u} \cdot \vec{v} = |\vec{u}||\vec{v}| \cos \theta}$$

where θ is the angle between $\vec{u}$ and $\vec{v}$ (with $0 \le \theta \le \pi$).

Key Fact:

The dot product is a **scalar**, not a vector!

In 2D:

$$\vec{u} \cdot \vec{v} = u_1 v_1 + u_2 v_2$$

Example 7: Computing Dot Products

Problem: Find the dot product:

(a) $\vec{u} = \langle 2, 3, -1 \rangle$ and $\vec{v} = \langle 1, -2, 4 \rangle$

(b) $\vec{a} = 3\vec{i} - 2\vec{j} + \vec{k}$ and $\vec{b} = \vec{i} + 4\vec{j} - 2\vec{k}$

Solution:

(a) Using the algebraic definition:

$$\begin{aligned} \vec{u} \cdot \vec{v} &= (2)(1) + (3)(-2) + (-1)(4) \\ &= 2 - 6 - 4 \\ &= -8 \end{aligned}$$

Answer (a): $\boxed{-8}$

(b) First write in component form:

$\vec{a} = \langle 3, -2, 1 \rangle$ and $\vec{b} = \langle 1, 4, -2 \rangle$

$$\begin{aligned} \vec{a} \cdot \vec{b} &= (3)(1) + (-2)(4) + (1)(-2) \\ &= 3 - 8 - 2 \\ &= -7 \end{aligned}$$

Answer (b): $\boxed{-7}$

Theorem 6.2: Properties of the Dot Product

Let $\vec{u}$, $\vec{v}$, $\vec{w}$ be vectors and c be a scalar.

1. $\vec{u} \cdot \vec{v} = \vec{v} \cdot \vec{u}$ (commutative)
2. $\vec{u} \cdot (\vec{v} + \vec{w}) = \vec{u} \cdot \vec{v} + \vec{u} \cdot \vec{w}$ (distributive)
3. $(c\vec{u}) \cdot \vec{v} = c(\vec{u} \cdot \vec{v}) = \vec{u} \cdot (c\vec{v})$
4. $\vec{v} \cdot \vec{v} = |\vec{v}|^2$ (magnitude squared)
5. $\vec{0} \cdot \vec{v} = 0$
6. $\vec{u} \cdot \vec{v} = 0$ if and only if $\vec{u} \perp \vec{v}$ or $\vec{u} = \vec{0}$ or $\vec{v} = \vec{0}$

Standard Unit Vectors:

$$\vec{i} \cdot \vec{i} = \vec{j} \cdot \vec{j} = \vec{k} \cdot \vec{k} = 1$$
$$\vec{i} \cdot \vec{j} = \vec{i} \cdot \vec{k} = \vec{j} \cdot \vec{k} = 0$$

Definition 6.9: Angle Between Vectors

The angle θ between two non-zero vectors $\vec{u}$ and $\vec{v}$ is found from:

$$\boxed{\cos\theta = \frac{\vec{u} \cdot \vec{v}}{|\vec{u}||\vec{v}|}}$$

Therefore:

$$\boxed{\theta = \arccos\left(\frac{\vec{u} \cdot \vec{v}}{|\vec{u}||\vec{v}|}\right)}$$

where $0 \leq \theta \leq \pi$.

Special Cases:

- $\theta = 0$: vectors point in same direction ($\vec{u} \cdot \vec{v} = |\vec{u}||\vec{v}|$)
- $\theta = \pi/2$: vectors are perpendicular ($\vec{u} \cdot \vec{v} = 0$)
- $\theta = \pi$: vectors point in opposite directions ($\vec{u} \cdot \vec{v} = -|\vec{u}||\vec{v}|$)

Important:
The angle between vectors is always between 0 and π (not 2π).

Example 8: Finding Angle Between Vectors

Problem: Find the angle between $\vec{u} = \langle 1, 2, 2\rangle$ and $\vec{v} = \langle 2, 1, -2\rangle$.

Solution:
Step 1: Compute the dot product

$$\vec{u} \cdot \vec{v} = (1)(2) + (2)(1) + (2)(-2) = 2 + 2 - 4 = 0$$

Step 2: Find magnitudes

$$|\vec{u}| = \sqrt{1^2 + 2^2 + 2^2} = \sqrt{9} = 3$$
$$|\vec{v}| = \sqrt{2^2 + 1^2 + (-2)^2} = \sqrt{9} = 3$$

Step 3: Apply the formula

$$\cos\theta = \frac{\vec{u}\cdot\vec{v}}{|\vec{u}||\vec{v}|} = \frac{0}{3\cdot 3} = 0$$

$$\theta = \arccos(0) = \frac{\pi}{2}$$

Answer: $\boxed{\theta = \frac{\pi}{2} = 90}$

Conclusion: The vectors are **perpendicular** (orthogonal).

Definition 6.10: Orthogonal Vectors

Two vectors $\vec{u}$ and $\vec{v}$ are **orthogonal** (perpendicular) if and only if:

$$\boxed{\vec{u}\cdot\vec{v} = 0}$$

Note:
The zero vector $\vec{0}$ is considered orthogonal to every vector.
Notation:
We write $\vec{u} \perp \vec{v}$ to indicate that $\vec{u}$ and $\vec{v}$ are orthogonal.

Example 9: Testing for Orthogonality

Problem: Determine if the following pairs of vectors are orthogonal:
(a) $\vec{u} = \langle 2, -3, 1\rangle$ and $\vec{v} = \langle 3, 2, 0\rangle$
(b) $\vec{a} = \langle 1, 1, 1\rangle$ and $\vec{b} = \langle 2, -1, -1\rangle$

Solution:
(a) Compute dot product:

$$\vec{u}\cdot\vec{v} = (2)(3) + (-3)(2) + (1)(0) = 6 - 6 + 0 = 0$$

Since $\vec{u}\cdot\vec{v} = 0$: $\boxed{\text{Yes, orthogonal}}$

(b) Compute dot product:

$$\vec{a}\cdot\vec{b} = (1)(2) + (1)(-1) + (1)(-1) = 2 - 1 - 1 = 0$$

Since $\vec{a}\cdot\vec{b} = 0$: $\boxed{\text{Yes, orthogonal}}$

Definition 6.11: Orthogonal Projection

The **orthogonal projection** of $\vec{u}$ onto $\vec{v}$ is the vector:

$$\boxed{\text{proj}_{\vec{v}}\vec{u} = \left(\frac{\vec{u}\cdot\vec{v}}{|\vec{v}|^2}\right)\vec{v} = \left(\frac{\vec{u}\cdot\vec{v}}{\vec{v}\cdot\vec{v}}\right)\vec{v}}$$

Scalar Component:
The scalar component of $\vec{u}$ in the direction of $\vec{v}$ is:

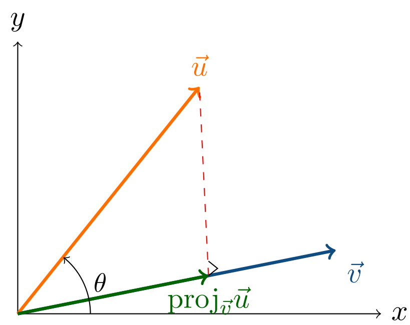
$$\boxed{\text{comp}_{\vec{v}}\vec{u} = \frac{\vec{u}\cdot\vec{v}}{|\vec{v}|} = |\vec{u}|\cos\theta}$$

Relationship:

$$\text{proj}_{\vec{v}}\vec{u} = (\text{comp}_{\vec{v}}\vec{u}) \cdot \frac{\vec{v}}{|\vec{v}|}$$

The projection is the component times the unit vector in direction of $\vec{v}$.

Geometric Interpretation:
$\text{proj}_{\vec{v}}\vec{u}$ is the "shadow" of $\vec{u}$ on $\vec{v}$.



Orthogonal Projection
$\text{proj}_{\vec{v}}\vec{u}$ is parallel to $\vec{v}$

Example 10: Computing Projections

Problem: Find the projection of $\vec{u} = \langle 3, 4, -1\rangle$ onto $\vec{v} = \langle 2, 1, 2\rangle$.

Solution:
Step 1: Compute $\vec{u}\cdot\vec{v}$

$$\vec{u}\cdot\vec{v} = (3)(2) + (4)(1) + (-1)(2) = 6 + 4 - 2 = 8$$

Step 2: Compute $|\vec{v}|^2$

$$|\vec{v}|^2 = 2^2 + 1^2 + 2^2 = 4 + 1 + 4 = 9$$

Step 3: Apply projection formula

$$\begin{aligned}\text{proj}_{\vec{v}}\vec{u} &= \frac{\vec{u}\cdot\vec{v}}{|\vec{v}|^2}\vec{v} \\ &= \frac{8}{9}\langle 2,1,2\rangle \\ &= \left\langle \frac{16}{9}, \frac{8}{9}, \frac{16}{9} \right\rangle\end{aligned}$$

Answer: $\boxed{\text{proj}_{\vec{v}}\vec{u} = \left\langle \frac{16}{9}, \frac{8}{9}, \frac{16}{9} \right\rangle}$

Scalar component:

$$\text{comp}_{\vec{v}}\vec{u} = \frac{\vec{u}\cdot\vec{v}}{|\vec{v}|} = \frac{8}{3}$$

Example 11: Application – Work

Problem: A force $\vec{F} = \langle 3,4,0\rangle$ N is applied to an object, causing it to move from point $A(1,1,0)$ to point $B(4,5,0)$. Find the work done.

Solution:

Step 1: Find displacement vector

$$\vec{d} = \overrightarrow{AB} = \langle 4-1, 5-1, 0-0\rangle = \langle 3,4,0\rangle$$

Step 2: Work formula
Work is defined as:

$$W = \vec{F}\cdot\vec{d}$$

Step 3: Compute

$$W = \langle 3,4,0\rangle\cdot\langle 3,4,0\rangle = 9+16+0 = 25$$

Answer: $\boxed{W = 25 \text{ J (joules)}}$

Physical Interpretation: The dot product gives the component of force in the direction of motion times the distance traveled.

Section 6.3 Summary: Dot Product

Dot Product Formula:

$$\boxed{\vec{u}\cdot\vec{v} = u_1v_1 + u_2v_2 + u_3v_3 = |\vec{u}||\vec{v}|\cos\theta}$$

Angle Between Vectors:

$$\boxed{\cos\theta = \frac{\vec{u}\cdot\vec{v}}{|\vec{u}||\vec{v}|}}$$

Orthogonality Test:

$$\boxed{\vec{u}\perp\vec{v} \iff \vec{u}\cdot\vec{v} = 0}$$

Projection:

$$\boxed{\text{proj}_{\vec{v}}\vec{u} = \left(\frac{\vec{u}\cdot\vec{v}}{|\vec{v}|^2}\right)\vec{v}}$$

Scalar Component:

$$\boxed{\text{comp}_{\vec{v}}\vec{u} = \frac{\vec{u} \cdot \vec{v}}{|\vec{v}|}}$$

Exam Alert: Orthogonal projections appear **frequently** in past papers!

⚠ Common Mistakes:

- Using $|\vec{v}|$ instead of $|\vec{v}|^2$ in projection formula
- Forgetting that dot product produces a scalar, not a vector
- Confusing projection (vector) with scalar component (number)
- Computing angle outside range $[0, \pi]$

Key Applications:

- Finding angles between vectors
- Testing for perpendicularity
- Computing projections
- Calculating work: $W = \vec{F} \cdot \vec{d}$

Cross Product and Applications

The cross product is an operation that takes two vectors in $\mathbb{R}^3$ and produces a third vector that is perpendicular to both. Unlike the dot product, the cross product produces a vector.

Definition 6.12: Cross Product

Given $\vec{u} = \langle u_1, u_2, u_3 \rangle$ and $\vec{v} = \langle v_1, v_2, v_3 \rangle$:

Algebraic Definition:

$$\boxed{\vec{u} \times \vec{v} = \langle u_2v_3 - u_3v_2, u_3v_1 - u_1v_3, u_1v_2 - u_2v_1 \rangle}$$

Determinant Form:

$$\boxed{\vec{u} \times \vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}}$$

Expand along the first row:

$$\vec{u} \times \vec{v} = \vec{i}\begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix} - \vec{j}\begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix} + \vec{k}\begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix}$$

⚠ **Important:**
The cross product is defined **only in** $\mathbb{R}^3$! It does not exist in 2D.

ℹ **Result:**
$\vec{u} \times \vec{v}$ is a **vector** (unlike dot product which is scalar).

Example 12: Computing Cross Product

Problem: Find $\vec{u} \times \vec{v}$ where $\vec{u} = \langle 2, 1, -1 \rangle$ and $\vec{v} = \langle 1, 3, 2 \rangle$.

Solution:

Method 1: Using the formula directly

$$\begin{aligned} \vec{u} \times \vec{v} &= \langle u_2v_3 - u_3v_2, u_3v_1 - u_1v_3, u_1v_2 - u_2v_1 \rangle \\ &= \langle (1)(2) - (-1)(3), (-1)(1) - (2)(2), (2)(3) - (1)(1) \rangle \\ &= \langle 2 + 3, -1 - 4, 6 - 1 \rangle \\ &= \langle 5, -5, 5 \rangle \end{aligned}$$

Method 2: Using determinant

$$\vec{u} \times \vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 1 & -1 \\ 1 & 3 & 2 \end{vmatrix}$$

$$\begin{aligned} &= \vec{i}\begin{vmatrix} 1 & -1 \\ 3 & 2 \end{vmatrix} - \vec{j}\begin{vmatrix} 2 & -1 \\ 1 & 2 \end{vmatrix} + \vec{k}\begin{vmatrix} 2 & 1 \\ 1 & 3 \end{vmatrix} \\ &= \vec{i}(2 + 3) - \vec{j}(4 + 1) + \vec{k}(6 - 1) \\ &= 5\vec{i} - 5\vec{j} + 5\vec{k} \\ &= \langle 5, -5, 5 \rangle \end{aligned}$$

Answer: $\boxed{\vec{u} \times \vec{v} = \langle 5, -5, 5 \rangle}$

Theorem 6.3: Properties of Cross Product

Let $\vec{u}$, $\vec{v}$, $\vec{w}$ be vectors and c be a scalar.

1. **Anti-commutative:**
$$\vec{u} \times \vec{v} = -(\vec{v} \times \vec{u})$$
2. **Distributive:**
$$\vec{u} \times (\vec{v} + \vec{w}) = \vec{u} \times \vec{v} + \vec{u} \times \vec{w}$$
3. **Scalar multiplication:**
$$(c\vec{u}) \times \vec{v} = c(\vec{u} \times \vec{v}) = \vec{u} \times (c\vec{v})$$
4. **Perpendicularity:**
$$\vec{u} \times \vec{v} \perp \vec{u} \quad \text{and} \quad \vec{u} \times \vec{v} \perp \vec{v}$$
5. **Parallel vectors:**
$$\vec{u} \times \vec{v} = \vec{0} \iff \vec{u} \text{ and } \vec{v} \text{ are parallel (or one is } \vec{0})$$
6. **Magnitude:**
$$|\vec{u} \times \vec{v}| = |\vec{u}||\vec{v}| \sin\theta$$

where θ is the angle between $\vec{u}$ and $\vec{v}$.

⚠ **Warning:**

The cross product is **NOT associative**: $\vec{u} \times (\vec{v} \times \vec{w}) \neq (\vec{u} \times \vec{v}) \times \vec{w}$ in general.

Example 13: Verifying Perpendicularity

Problem: Verify that $\vec{u} \times \vec{v}$ from Example 12 is perpendicular to both $\vec{u}$ and $\vec{v}$.

Solution:

From Example 12: $\vec{u} = \langle 2, 1, -1 \rangle$, $\vec{v} = \langle 1, 3, 2 \rangle$, $\vec{u} \times \vec{v} = \langle 5, -5, 5 \rangle$

Check 1: $(\vec{u} \times \vec{v}) \cdot \vec{u}$
$$\langle 5, -5, 5 \rangle \cdot \langle 2, 1, -1 \rangle = 10 - 5 - 5 = 0 \quad \checkmark$$
Check 2: $(\vec{u} \times \vec{v}) \cdot \vec{v}$
$$\langle 5, -5, 5 \rangle \cdot \langle 1, 3, 2 \rangle = 5 - 15 + 10 = 0 \quad \checkmark$$
Conclusion: $\vec{u} \times \vec{v}$ is perpendicular to both $\vec{u}$ and $\vec{v}$.

Definition 6.13: Geometric Interpretation of Cross Product

Direction: $\vec{u} \times \vec{v}$ is perpendicular to both $\vec{u}$ and $\vec{v}$, following the **right-hand rule**:

- Point fingers of right hand along $\vec{u}$
- Curl them toward $\vec{v}$

- Thumb points in direction of $\vec{u} \times \vec{v}$

Magnitude:

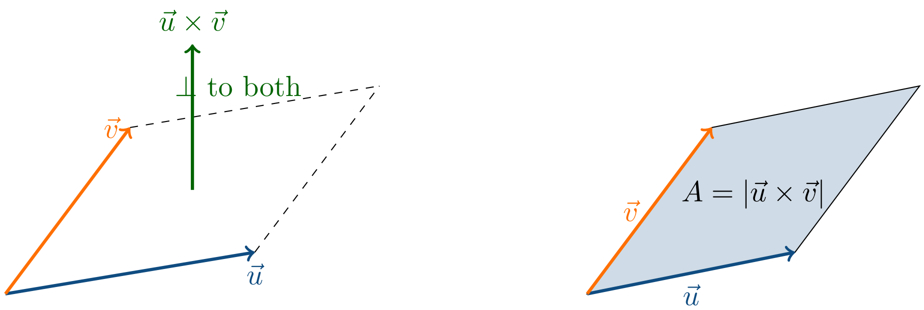
$$|\vec{u} \times \vec{v}| = |\vec{u}||\vec{v}| \sin\theta$$

This equals the area of the parallelogram with sides $\vec{u}$ and $\vec{v}$.

Area Interpretation:

$$\text{Area of parallelogram} = |\vec{u} \times \vec{v}|$$

$$\text{Area of triangle} = \frac{1}{2}|\vec{u} \times \vec{v}|$$



Cross Product Direction **Area = Magnitude**

Example 14: Finding Area of Parallelogram

Problem: Find the area of the parallelogram with vertices at $P(1, 0, 1)$, $Q(2, 3, 1)$, $R(3, 5, 2)$, and $S(2, 2, 2)$.

Solution:

Step 1: Find two adjacent sides as vectors

$$\overrightarrow{PQ} = \langle 2-1, 3-0, 1-1 \rangle = \langle 1, 3, 0 \rangle$$
$$\overrightarrow{PS} = \langle 2-1, 2-0, 2-1 \rangle = \langle 1, 2, 1 \rangle$$

Step 2: Compute cross product

$$\overrightarrow{PQ} \times \overrightarrow{PS} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 3 & 0 \\ 1 & 2 & 1 \end{vmatrix}$$

$$= \vec{i}(3-0) - \vec{j}(1-0) + \vec{k}(2-3)$$
$$= 3\vec{i} - \vec{j} - \vec{k}$$
$$= \langle 3, -1, -1 \rangle$$

Step 3: Find magnitude

$$|\overrightarrow{PQ} \times \overrightarrow{PS}| = \sqrt{3^2 + (-1)^2 + (-1)^2} = \sqrt{9+1+1} = \sqrt{11}$$

Answer: $\boxed{\text{Area} = \sqrt{11} \text{ square units}}$

Definition 6.14: Scalar Triple Product

The **scalar triple product** of vectors $\vec{u}$, $\vec{v}$, $\vec{w}$ is:

$$\boxed{\vec{u}\cdot(\vec{v}\times\vec{w})}$$

This can be computed as a determinant:

$$\vec{u}\cdot(\vec{v}\times\vec{w}) = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$$

Geometric Interpretation:
$|\vec{u}\cdot(\vec{v}\times\vec{w})|$ is the volume of the parallelepiped with edges $\vec{u}$, $\vec{v}$, $\vec{w}$.
Properties:

- $\vec{u}\cdot(\vec{v}\times\vec{w}) = \vec{v}\cdot(\vec{w}\times\vec{u}) = \vec{w}\cdot(\vec{u}\times\vec{v})$ (cyclic)
- $\vec{u}\cdot(\vec{v}\times\vec{w}) = 0 \iff$ vectors are coplanar

Example 15: Volume Using Scalar Triple Product

Problem: Find the volume of the parallelepiped determined by $\vec{u} = \langle 1,2,0\rangle$, $\vec{v} = \langle 0,1,3\rangle$, and $\vec{w} = \langle 2,0,1\rangle$.

Solution:
Step 1: Compute scalar triple product using determinant

$$\vec{u}\cdot(\vec{v}\times\vec{w}) = \begin{vmatrix} 1 & 2 & 0 \\ 0 & 1 & 3 \\ 2 & 0 & 1 \end{vmatrix}$$

Step 2: Expand (using first row)

$$\begin{aligned} &= 1\begin{vmatrix} 1 & 3 \\ 0 & 1 \end{vmatrix} - 2\begin{vmatrix} 0 & 3 \\ 2 & 1 \end{vmatrix} + 0\begin{vmatrix} 0 & 1 \\ 2 & 0 \end{vmatrix} \\ &= 1(1-0) - 2(0-6) + 0 \\ &= 1 + 12 \\ &= 13 \end{aligned}$$

Step 3: Volume is absolute value

$$V = |\vec{u}\cdot(\vec{v}\times\vec{w})| = |13| = 13$$

Answer: $\boxed{V = 13 \text{ cubic units}}$

Section 6.4 Summary: Cross Product

Cross Product Formula:

$$\boxed{\vec{u}\times\vec{v}=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ u_1 & u_2 & u_3\\ v_1 & v_2 & v_3\end{vmatrix}}$$

Key Properties:

- $\vec{u}\times\vec{v}=-\vec{v}\times\vec{u}$ (anti-commutative)
- $\vec{u}\times\vec{v}\perp\vec{u}$ and $\vec{u}\times\vec{v}\perp\vec{v}$
- $|\vec{u}\times\vec{v}|=|\vec{u}||\vec{v}|\sin\theta$
- $\vec{u}\times\vec{v}=\vec{0}\iff\vec{u}\parallel\vec{v}$

Applications:

$$\begin{aligned}\text{Area of parallelogram: }\quad & |\vec{u}\times\vec{v}|\\ \text{Area of triangle: }\quad & \frac{1}{2}|\vec{u}\times\vec{v}|\\ \text{Volume of parallelepiped: }\quad & |\vec{u}\cdot(\vec{v}\times\vec{w})|\end{aligned}$$

Standard Cross Products:

$$\begin{aligned}&\vec{i}\times\vec{j}=\vec{k},\quad \vec{j}\times\vec{k}=\vec{i},\quad \vec{k}\times\vec{i}=\vec{j}\\ &\vec{j}\times\vec{i}=-\vec{k},\quad \vec{k}\times\vec{j}=-\vec{i},\quad \vec{i}\times\vec{k}=-\vec{j}\end{aligned}$$

⚠ **Common Mistakes:**

- Forgetting the minus sign on the $\vec{j}$ component in determinant expansion
- Thinking cross product is commutative (it's anti-commutative!)
- Trying to use cross product in 2D (only defined in 3D)
- Confusing $\vec{u}\times\vec{v}$ with $\vec{u}\cdot\vec{v}$ (vector vs. scalar)

Quick Check:

- $\vec{i}\times\vec{j}=\vec{k}$ (right-hand rule)
- Cross product result is perpendicular to both original vectors
- Magnitude equals area of parallelogram

Vector Functions and Parametric Curves

Vector functions provide a way to describe curves in space parametrically. Instead of expressing y as a function of x, we express both x, y, and z as functions of a parameter t.

Definition 6.15: Vector Function

A **vector function** (or **vector-valued function**) is a function that assigns a vector to each value of a parameter t in its domain.

In 2D:

$$\vec{r}(t) = \langle f(t), g(t) \rangle = f(t)\vec{i} + g(t)\vec{j}$$

In 3D:

$$\boxed{\vec{r}(t) = \langle f(t), g(t), h(t) \rangle = f(t)\vec{i} + g(t)\vec{j} + h(t)\vec{k}}$$

where f, g, and h are called the **component functions**.

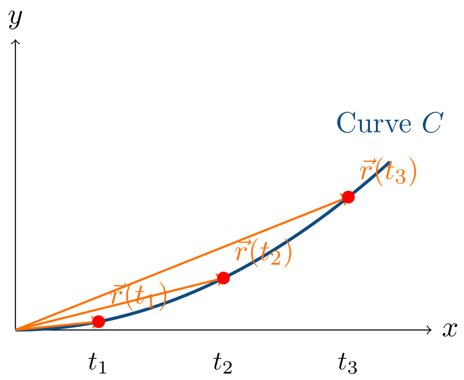
Geometric Interpretation:

As t varies, the tip of the vector $\vec{r}(t)$ traces out a curve C in space. The curve is called the **graph** of $\vec{r}(t)$.

Parametric Equations:

The component form gives the parametric equations:

$$x = f(t), \quad y = g(t), \quad z = h(t)$$



Vector Function Traces a Curve

Example 16: Graphing a Vector Function

Problem: Describe the curve traced by $\vec{r}(t) = \langle \cos t, \sin t, t \rangle$ for $t \geq 0$.

Solution:

Step 1: Identify parametric equations

$$x = \cos t$$
$$y = \sin t$$
$$z = t$$

Step 2: Analyze the xy-projection

From $x^2 + y^2 = \cos^2 t + \sin^2 t = 1$, the projection onto the xy-plane is a circle of radius 1.

Step 3: Include the z-component
As t increases from 0 to ∞:

- (x, y) traces around the unit circle
- $z = t$ increases linearly

Answer: The curve is a **circular helix** (spiral) that winds around the z-axis.
Key points:

- At $t = 0$: $(1, 0, 0)$
- At $t = \pi/2$: $(0, 1, \pi/2)$
- At $t = \pi$: $(-1, 0, \pi)$
- At $t = 2\pi$: $(1, 0, 2\pi)$

Visualization: The helix rises while circling the z-axis with constant radius 1.

Example 17: Finding a Vector Function for a Curve

Problem: Find a vector function that represents the curve of intersection of:

- Cylinder: $x^2 + y^2 = 4$
- Plane: $z = x + y$

Solution:
Step 1: Parametrize the cylinder
Use $x = 2\cos t$ and $y = 2\sin t$ (circle of radius 2).
Step 2: Find z from the plane equation

$$\begin{aligned} z &= x + y \\ &= 2\cos t + 2\sin t \\ &= 2(\cos t + \sin t) \end{aligned}$$

Step 3: Write vector function

$$\vec{r}(t) = \langle 2\cos t, 2\sin t, 2(\cos t + \sin t)\rangle$$

for $0 \leq t \leq 2\pi$.
Answer: $\boxed{\vec{r}(t) = \langle 2\cos t, 2\sin t, 2\cos t + 2\sin t\rangle}$

Definition 6.16: Limits and Continuity

Limit of a Vector Function:

$$\boxed{\lim_{t\to a} \vec{r}(t) = \left\langle \lim_{t\to a} f(t), \lim_{t\to a} g(t), \lim_{t\to a} h(t) \right\rangle}$$

provided each component limit exists.
Continuity:

$\vec{r}(t)$ is **continuous** at $t = a$ if:

$$\lim_{t \to a} \vec{r}(t) = \vec{r}(a)$$

This means each component function is continuous at a.

Domain:

The domain of $\vec{r}(t)$ is the intersection of the domains of its component functions.

Example 18: Limits and Continuity

Problem: Find $\lim_{t \to 0} \vec{r}(t)$ where $\vec{r}(t) = \left\langle \frac{\sin t}{t}, t^2 - 1, e^t \right\rangle$.

Solution:

Compute the limit of each component:

Component 1:

$$\lim_{t \to 0} \frac{\sin t}{t} = 1$$

Component 2:

$$\lim_{t \to 0} (t^2 - 1) = 0 - 1 = -1$$

Component 3:

$$\lim_{t \to 0} e^t = e^0 = 1$$

Answer: $\boxed{\lim_{t \to 0} \vec{r}(t) = \langle 1, -1, 1 \rangle}$

Definition 6.17: Common Space Curves

1. Line:

$$\vec{r}(t) = \vec{r}_0 + t\vec{v}$$

where $\vec{r}_0$ is a point on the line and $\vec{v}$ is the direction vector.

2. Circle (in xy-plane):

$$\vec{r}(t) = \langle a \cos t, a \sin t, 0 \rangle$$

Circle of radius a centered at origin.

3. Helix:

$$\vec{r}(t) = \langle a \cos t, a \sin t, bt \rangle$$

Circular helix with radius a and pitch determined by b.

4. Ellipse:

$$\vec{r}(t) = \langle a \cos t, b \sin t, 0 \rangle$$

Ellipse with semi-axes a and b.

5. Parabola:

$$\vec{r}(t) = \langle t, t^2, 0 \rangle$$

Standard parabola in xy-plane.

Section 6.5 Summary: Vector Functions

Vector Function:

$$\boxed{\vec{r}(t) = \langle f(t), g(t), h(t) \rangle}$$

Parametric Equations:

$$x = f(t), \quad y = g(t), \quad z = h(t)$$

Limit:

$$\lim_{t \to a} \vec{r}(t) = \left\langle \lim_{t \to a} f(t), \lim_{t \to a} g(t), \lim_{t \to a} h(t) \right\rangle$$

Common Curves:

- Line: $\vec{r}(t) = \vec{r}_0 + t\vec{v}$
- Circle: $\vec{r}(t) = \langle a \cos t, a \sin t, 0 \rangle$
- Helix: $\vec{r}(t) = \langle a \cos t, a \sin t, bt \rangle$

Common Mistakes:

- Confusing vector function with scalar function
- Not checking domain restrictions for each component
- Forgetting that limits are computed component-wise

Key Insight: Vector functions parametrize curves in space, giving position as a function of time or parameter.

Derivatives and Integrals of Vector Functions

Just as we can differentiate and integrate scalar functions, we can perform these operations on vector functions by working component-wise.

Definition 6.18: Derivative of a Vector Function

The **derivative** of $\vec{r}(t) = \langle f(t), g(t), h(t)\rangle$ is:

$$\boxed{\vec{r}'(t) = \lim_{\Delta t \to 0} \frac{\vec{r}(t+\Delta t) - \vec{r}(t)}{\Delta t}}$$

Component-wise:

$$\boxed{\vec{r}'(t) = \langle f'(t), g'(t), h'(t)\rangle}$$

Alternative Notations:

$$\vec{r}'(t) = \frac{d\vec{r}}{dt} = \dot{\vec{r}}(t)$$

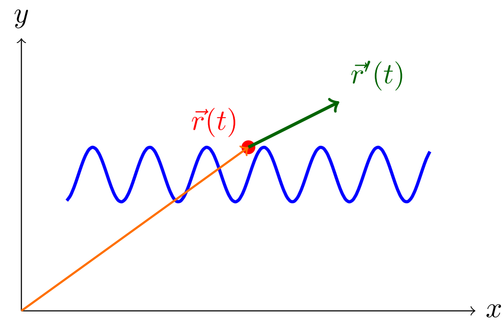
Geometric Interpretation:

$\vec{r}'(t)$ is the **tangent vector** to the curve at the point $\vec{r}(t)$. It points in the direction of motion along the curve.

Physical Interpretation:

If $\vec{r}(t)$ represents position at time t, then:

- $\vec{r}'(t) = \vec{v}(t)$ is the velocity vector
- $|\vec{r}'(t)|$ is the speed



Tangent Vector
$\vec{r}'(t)$ points in direction of motion

Example 19: Computing Derivatives

Problem: Find $\vec{r}'(t)$ and $\vec{r}''(t)$ for $\vec{r}(t) = \langle t^2, e^t, \sin t\rangle$.

Solution:

First derivative: Differentiate each component

$$\vec{r}'(t) = \langle 2t, e^t, \cos t\rangle$$

Second derivative: Differentiate $\vec{r}'(t)$

$$\vec{r}''(t) = \langle 2, e^t, -\sin t\rangle$$

Answer:

$$\vec{r}'(t) = \boxed{\langle 2t, e^t, \cos t\rangle}$$
$$\vec{r}''(t) = \boxed{\langle 2, e^t, -\sin t\rangle}$$

Interpretation: $\vec{r}'(t)$ is velocity, $\vec{r}''(t)$ is acceleration.

Theorem 6.4: Differentiation Rules for Vector Functions

Let $\vec{u}(t)$ and $\vec{v}(t)$ be differentiable vector functions, c a scalar, and $f(t)$ a scalar function.

1. $\dfrac{d}{dt}[c\vec{u}(t)] = c\vec{u}'(t)$
2. $\dfrac{d}{dt}[\vec{u}(t) + \vec{v}(t)] = \vec{u}'(t) + \vec{v}'(t)$
3. $\dfrac{d}{dt}[f(t)\vec{u}(t)] = f'(t)\vec{u}(t) + f(t)\vec{u}'(t)$ (product rule)
4. $\dfrac{d}{dt}[\vec{u}(t) \cdot \vec{v}(t)] = \vec{u}'(t) \cdot \vec{v}(t) + \vec{u}(t) \cdot \vec{v}'(t)$ (dot product)
5. $\dfrac{d}{dt}[\vec{u}(t) \times \vec{v}(t)] = \vec{u}'(t) \times \vec{v}(t) + \vec{u}(t) \times \vec{v}'(t)$ (cross product)
6. $\dfrac{d}{dt}[\vec{u}(f(t))] = \vec{u}'(f(t))f'(t)$ (chain rule)

Warning:
In rule 5, order matters for cross products! Don't switch the order of $\vec{u}'(t) \times \vec{v}(t)$ and $\vec{u}(t) \times \vec{v}'(t)$.

Example 20: Using Differentiation Rules

Problem: If $\vec{r}(t) = \langle \cos t, \sin t, t\rangle$, show that $\vec{r}(t) \cdot \vec{r}'(t) = t$.

Solution:
Step 1: Find $\vec{r}'(t)$

$$\vec{r}'(t) = \langle -\sin t, \cos t, 1\rangle$$

Step 2: Compute dot product

$$\begin{aligned}
\vec{r}(t) \cdot \vec{r}'(t) &= \langle \cos t, \sin t, t\rangle \cdot \langle -\sin t, \cos t, 1\rangle \\
&= (\cos t)(-\sin t) + (\sin t)(\cos t) + t(1) \\
&= -\sin t \cos t + \sin t \cos t + t \\
&= t
\end{aligned}$$

Answer: $\vec{r}(t) \cdot \vec{r}'(t) = t$ ✓

Alternative using product rule:

$$\begin{aligned}
\frac{d}{dt}[\vec{r}(t)\cdot\vec{r}(t)] &= 2\vec{r}(t)\cdot\vec{r}'(t)\\
\frac{d}{dt}[|\vec{r}(t)|^2] &= 2\vec{r}(t)\cdot\vec{r}'(t)\\
\frac{d}{dt}[\cos^2 t+\sin^2 t+t^2] &= 2\vec{r}(t)\cdot\vec{r}'(t)\\
2t &= 2\vec{r}(t)\cdot\vec{r}'(t)\\
t &= \vec{r}(t)\cdot\vec{r}'(t)
\end{aligned}$$

Definition 6.19: Unit Tangent Vector

The **unit tangent vector** to a curve at a point is:

$$\boxed{\vec{T}(t)=\frac{\vec{r}'(t)}{|\vec{r}'(t)|}}$$

This is a unit vector in the direction of $\vec{r}'(t)$.

Properties:

- $|\vec{T}(t)|=1$ (unit vector)
- $\vec{T}(t)$ is tangent to the curve
- Points in direction of increasing t

Example 21: Finding Unit Tangent Vector

Problem: Find the unit tangent vector to $\vec{r}(t)=\langle t^2,2t,\ln t\rangle$ at $t=1$.

Solution:

Step 1: Find $\vec{r}'(t)$

$$\vec{r}'(t)=\left\langle 2t,2,\frac{1}{t}\right\rangle$$

Step 2: Evaluate at $t=1$

$$\vec{r}'(1)=\langle 2,2,1\rangle$$

Step 3: Find magnitude

$$|\vec{r}'(1)|=\sqrt{4+4+1}=\sqrt{9}=3$$

Step 4: Compute unit tangent vector

$$\vec{T}(1)=\frac{\vec{r}'(1)}{|\vec{r}'(1)|}=\frac{1}{3}\langle 2,2,1\rangle=\left\langle\frac{2}{3},\frac{2}{3},\frac{1}{3}\right\rangle$$

Answer: $\boxed{\vec{T}(1)=\left\langle\frac{2}{3},\frac{2}{3},\frac{1}{3}\right\rangle}$

Definition 6.20: Integral of a Vector Function

The **indefinite integral** of $\vec{r}(t) = \langle f(t), g(t), h(t)\rangle$ is:

$$\boxed{\int \vec{r}(t)\,dt = \left\langle \int f(t)\,dt, \int g(t)\,dt, \int h(t)\,dt \right\rangle + \vec{C}}$$

where $\vec{C}$ is a constant vector.
The **definite integral** is:

$$\boxed{\int_a^b \vec{r}(t)\,dt = \left\langle \int_a^b f(t)\,dt, \int_a^b g(t)\,dt, \int_a^b h(t)\,dt \right\rangle}$$

Key Point:
Integration is done component-wise, just like differentiation.
Fundamental Theorem:

$$\int_a^b \vec{r}\,'(t)\,dt = \vec{r}(b) - \vec{r}(a)$$

Example 22: Integrating Vector Functions

Problem: Find:
(a) $\int \langle 2t, \cos t, e^t\rangle\,dt$
(b) $\int_0^1 \langle t^2, t, 1\rangle\,dt$

Solution:
(a) Integrate each component:

$$\begin{aligned}\int \langle 2t, \cos t, e^t\rangle\,dt &= \left\langle \int 2t\,dt, \int \cos t\,dt, \int e^t\,dt \right\rangle \\ &= \langle t^2, \sin t, e^t\rangle + \vec{C}\end{aligned}$$

Answer (a): $\boxed{\langle t^2, \sin t, e^t\rangle + \vec{C}}$

(b) Integrate and evaluate:

$$\begin{aligned}\int_0^1 \langle t^2, t, 1\rangle\,dt &= \left\langle \int_0^1 t^2\,dt, \int_0^1 t\,dt, \int_0^1 1\,dt \right\rangle \\ &= \left\langle \left[\frac{t^3}{3}\right]_0^1, \left[\frac{t^2}{2}\right]_0^1, [t]_0^1 \right\rangle \\ &= \left\langle \frac{1}{3}, \frac{1}{2}, 1 \right\rangle\end{aligned}$$

Answer (b): $\boxed{\left\langle \frac{1}{3}, \frac{1}{2}, 1 \right\rangle}$

Example 23: Initial Value Problem

Problem: Find $\vec{r}(t)$ if $\vec{r}'(t) = \langle 6t, 3t^2, 2\rangle$ and $\vec{r}(0) = \langle 1, -1, 3\rangle$.

Solution:
Step 1: Integrate $\vec{r}'(t)$

$$\begin{aligned}\vec{r}(t) &= \int \langle 6t, 3t^2, 2\rangle\, dt \\ &= \langle 3t^2, t^3, 2t\rangle + \vec{C}\end{aligned}$$

where $\vec{C} = \langle C_1, C_2, C_3\rangle$ is a constant vector.
Step 2: Apply initial condition

$$\vec{r}(0) = \langle 0, 0, 0\rangle + \vec{C} = \langle 1, -1, 3\rangle$$

Therefore: $\vec{C} = \langle 1, -1, 3\rangle$
Step 3: Write final answer

$$\vec{r}(t) = \langle 3t^2, t^3, 2t\rangle + \langle 1, -1, 3\rangle = \langle 3t^2 + 1, t^3 - 1, 2t + 3\rangle$$

Answer: $\boxed{\vec{r}(t) = \langle 3t^2 + 1, t^3 - 1, 2t + 3\rangle}$
Verification:

- $\vec{r}'(t) = \langle 6t, 3t^2, 2\rangle$ ✓
- $\vec{r}(0) = \langle 1, -1, 3\rangle$ ✓

Section 6.6 Summary: Derivatives and Integrals

Derivative:

$$\boxed{\vec{r}'(t) = \langle f'(t), g'(t), h'(t)\rangle}$$

Unit Tangent Vector:

$$\boxed{\vec{T}(t) = \frac{\vec{r}'(t)}{|\vec{r}'(t)|}}$$

Indefinite Integral:

$$\boxed{\int \vec{r}(t)\, dt = \left\langle \int f(t)\, dt, \int g(t)\, dt, \int h(t)\, dt \right\rangle + \vec{C}}$$

Definite Integral:

$$\boxed{\int_a^b \vec{r}(t)\, dt = \left\langle \int_a^b f(t)\, dt, \int_a^b g(t)\, dt, \int_a^b h(t)\, dt \right\rangle}$$

Product Rules:

$$\begin{aligned}\frac{d}{dt}[\vec{u} \cdot \vec{v}] &= \vec{u}' \cdot \vec{v} + \vec{u} \cdot \vec{v}' \\ \frac{d}{dt}[\vec{u} \times \vec{v}] &= \vec{u}' \times \vec{v} + \vec{u} \times \vec{v}'\end{aligned}$$

Physical Interpretations:

- $\vec{r}(t)$: position
- $\vec{r}'(t) = \vec{v}(t)$: velocity
- $\vec{r}''(t) = \vec{a}(t)$: acceleration
- $|\vec{r}'(t)|$: speed

⚠ **Common Mistakes:**

- Forgetting to include constant vector $\vec{C}$ in indefinite integrals
- Not differentiating/integrating all components
- Order errors in cross product differentiation
- Forgetting that $\vec{T}(t)$ must be a unit vector

Key Insight: Differentiation and integration of vector functions work component-wise, just like scalar functions.

Lines in $\mathbb{R}^3$

Lines in three-dimensional space can be described using vector equations, parametric equations, or symmetric equations. Understanding these different forms is essential for working with 3D geometry.

Definition 6.21: Vector Equation of a Line

A line L in $\mathbb{R}^3$ that passes through point $P_0(x_0, y_0, z_0)$ and is parallel to vector $\vec{v} = \langle a, b, c\rangle$ has **vector equation**:

$$\boxed{\vec{r}(t) = \vec{r}_0 + t\vec{v}}$$

where $\vec{r}_0 = \langle x_0, y_0, z_0\rangle$ is the position vector of P_0.
Equivalently:

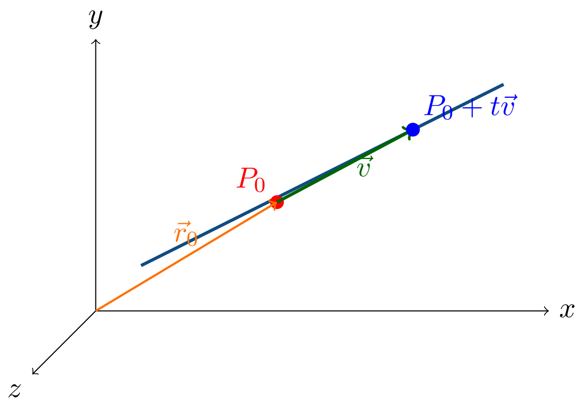
$$\boxed{\vec{r}(t) = \langle x_0, y_0, z_0\rangle + t\langle a, b, c\rangle}$$

Components:

- $\vec{r}_0$: position vector of a point on the line
- $\vec{v}$: direction vector (parallel to the line)
- t: parameter ($-\infty < t < \infty$)

Note:
Different values of t give different points on the line:

- $t = 0$: gives point P_0
- $t > 0$: points in direction of $\vec{v}$
- $t < 0$: points in opposite direction



Line: $\vec{r}(t) = \vec{r}_0 + t\vec{v}$

Definition 6.22: Parametric Equations of a Line

The **parametric equations** of the line are obtained by writing out components:

$$\boxed{\begin{cases} x = x_0 + at \\ y = y_0 + bt \\ z = z_0 + ct \end{cases}}$$

where (x_0, y_0, z_0) is a point on the line and $\langle a, b, c\rangle$ is the direction vector.

Interpretation:

Each coordinate is a linear function of the parameter t.

Example 24: Finding Parametric Equations

Problem: Find parametric equations for the line through $P(1, 2, -1)$ parallel to $\vec{v} = \langle 3, -2, 4\rangle$.

Solution:

Given:

- Point: $(x_0, y_0, z_0) = (1, 2, -1)$
- Direction: $\langle a, b, c\rangle = \langle 3, -2, 4\rangle$

Parametric equations:

$$\begin{cases} x = 1 + 3t \\ y = 2 - 2t \\ z = -1 + 4t \end{cases}$$

Vector form:

$$\vec{r}(t) = \langle 1, 2, -1\rangle + t\langle 3, -2, 4\rangle$$

Answer: $\boxed{\begin{cases} x = 1 + 3t \\ y = 2 - 2t \\ z = -1 + 4t \end{cases}}$

Definition 6.23: Symmetric Equations of a Line

If $a \neq 0$, $b \neq 0$, and $c \neq 0$, we can solve the parametric equations for t and equate:

$$\boxed{\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}}$$

These are the **symmetric equations** of the line.

Special Cases:

If one direction component is zero (e.g., $a = 0$), write:

$$x = x_0, \quad \frac{y - y_0}{b} = \frac{z - z_0}{c}$$

Advantage:

Symmetric form doesn't explicitly show the parameter t.

Disadvantage:
Cannot be used if any direction component is zero (without modification).

Example 25: Line Through Two Points

Problem: Find parametric and symmetric equations for the line through $P(2, 1, 0)$ and $Q(4, -1, 3)$.

Solution:
Step 1: Find direction vector

$$\vec{v} = \overrightarrow{PQ} = \langle 4-2, -1-1, 3-0\rangle = \langle 2, -2, 3\rangle$$

Step 2: Use point P and direction $\vec{v}$ for parametric equations

$$\begin{cases} x = 2 + 2t \\ y = 1 - 2t \\ z = 0 + 3t \end{cases}$$

Step 3: Convert to symmetric form
Solve each equation for t:

$$t = \frac{x-2}{2}$$
$$t = \frac{y-1}{-2}$$
$$t = \frac{z-0}{3}$$

Equate:

$$\frac{x-2}{2} = \frac{y-1}{-2} = \frac{z}{3}$$

Answer:

- Parametric: $\boxed{\begin{cases} x = 2 + 2t \\ y = 1 - 2t \\ z = 3t \end{cases}}$
- Symmetric: $\boxed{\frac{x-2}{2} = \frac{y-1}{-2} = \frac{z}{3}}$

Definition 6.24: Parallel and Perpendicular Lines

Two lines L_1 and L_2 with direction vectors $\vec{v}_1$ and $\vec{v}_2$ are:
Parallel: if $\vec{v}_1 = k\vec{v}_2$ for some scalar k

$$L_1 \parallel L_2 \iff \vec{v}_1 \parallel \vec{v}_2$$

Perpendicular: if $\vec{v}_1 \cdot \vec{v}_2 = 0$

$$L_1 \perp L_2 \iff \vec{v}_1 \perp \vec{v}_2$$

Skew Lines:

In $\mathbb{R}^3$, two lines can be:

- Intersecting (cross at a point)
- Parallel (same direction, don't meet)
- Skew (not parallel, don't intersect)

Skew lines are lines that are not parallel but don't intersect (they "miss" each other in 3D space).

Example 26: Testing for Parallel Lines

Problem: Determine if the lines are parallel:

$$\begin{aligned} L_1 : &\quad x = 1 + 2t, \quad y = -1 + 3t, \quad z = 2 - t \\ L_2 : &\quad x = 3 - 4s, \quad y = 2 - 6s, \quad z = 1 + 2s \end{aligned}$$

Solution:
Step 1: Identify direction vectors

$$\begin{aligned} \vec{v}_1 &= \langle 2, 3, -1 \rangle \\ \vec{v}_2 &= \langle -4, -6, 2 \rangle \end{aligned}$$

Step 2: Check if one is a scalar multiple of the other

$$\vec{v}_2 = -2\vec{v}_1 = -2\langle 2, 3, -1 \rangle = \langle -4, -6, 2 \rangle \quad \checkmark$$

Answer: $\boxed{\text{Yes, the lines are parallel}}$
Since $\vec{v}_2 = -2\vec{v}_1$, the direction vectors are parallel.

Example 27: Finding Intersection Point

Problem: Determine if the lines intersect, and if so, find the point of intersection:

$$\begin{aligned} L_1 : &\quad x = 1 + t, \quad y = 2 + 2t, \quad z = 1 - t \\ L_2 : &\quad x = 3 - s, \quad y = 1 + s, \quad z = s \end{aligned}$$

Solution:
Step 1: Set coordinates equal (different parameters!)

$$\begin{aligned} 1 + t &= 3 - s \quad \text{(x-coordinates)} \\ 2 + 2t &= 1 + s \quad \text{(y-coordinates)} \\ 1 - t &= s \quad \text{(z-coordinates)} \end{aligned}$$

Step 2: Solve the system
From equation 3: $s = 1 - t$

Substitute into equation 1:

$$\begin{aligned} 1+t &= 3-(1-t) \\ 1+t &= 2+t \\ 1 &= 2 \quad \text{(contradiction!)} \end{aligned}$$

Answer: $\boxed{\text{The lines do not intersect (they are skew)}}$

Note: Since we got a contradiction, the lines are skew (they don't meet and aren't parallel).

Section 6.7 Summary: Lines in $\mathbb{R}^3$

Vector Form:

$$\boxed{\vec{r}(t) = \vec{r}_0 + t\vec{v}}$$

Parametric Form:

$$\boxed{\begin{cases} x = x_0 + at \\ y = y_0 + bt \\ z = z_0 + ct \end{cases}}$$

Symmetric Form:

$$\boxed{\frac{x-x_0}{a} = \frac{y-y_0}{b} = \frac{z-z_0}{c}}$$

(provided $a, b, c \neq 0$)

Line Through Two Points:

Direction vector: $\vec{v} = \overrightarrow{PQ} = \langle x_2 - x_1, y_2 - y_1, z_2 - z_1 \rangle$

Parallel Lines: $\vec{v}_1 = k\vec{v}_2$

Perpendicular Lines: $\vec{v}_1 \cdot \vec{v}_2 = 0$

Common Mistakes:

- Using the same parameter for both lines when testing intersection
- Forgetting to check all three coordinate equations
- Trying to use symmetric form when a direction component is zero
- Concluding lines don't intersect when they're actually parallel

Testing Intersection: Set parametric equations equal (use different parameters!) and solve. If solution exists and satisfies all three equations, lines intersect.

Planes in $\mathbb{R}^3$

A plane in three-dimensional space is uniquely determined by a point and a normal vector (a vector perpendicular to the plane). Understanding plane equations is crucial for 3D geometry.

Definition 6.25: Equation of a Plane

A plane in $\mathbb{R}^3$ that contains point $P_0(x_0, y_0, z_0)$ and has **normal vector** $\vec{n} = \langle a, b, c\rangle$ has equation:

Vector Form:

$$\boxed{\vec{n} \cdot (\vec{r} - \vec{r}_0) = 0}$$

Scalar Form (Standard Equation):

$$\boxed{a(x - x_0) + b(y - y_0) + c(z - z_0) = 0}$$

General Form:

$$\boxed{ax + by + cz + d = 0}$$

where $d = -(ax_0 + by_0 + cz_0)$.

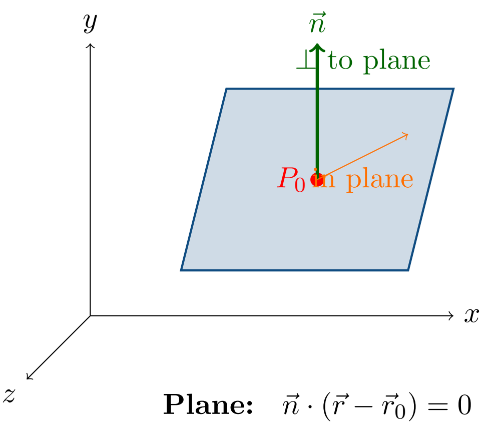
Key Concept:

The normal vector $\vec{n} = \langle a, b, c\rangle$ is perpendicular to every vector in the plane.

Derivation:

If $P(x, y, z)$ is any point on the plane, then $\overrightarrow{P_0P}$ lies in the plane, so:

$$\vec{n} \cdot \overrightarrow{P_0P} = 0$$



Plane: $\vec{n} \cdot (\vec{r} - \vec{r}_0) = 0$

Example 28: Finding Equation of a Plane

Problem: Find an equation of the plane through $P(1, 2, 3)$ with normal vector $\vec{n} = \langle 2, -1, 4\rangle$.

Solution:

Method 1: Scalar form

Using $a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$:

$$2(x - 1) + (-1)(y - 2) + 4(z - 3) = 0$$

Expand:

$$\begin{aligned} 2x - 2 - y + 2 + 4z - 12 &= 0 \\ 2x - y + 4z - 12 &= 0 \end{aligned}$$

Method 2: General form directly
$d = -(2 \cdot 1 + (-1) \cdot 2 + 4 \cdot 3) = -(2 - 2 + 12) = -12$
So: $2x - y + 4z - 12 = 0$
Answer: $\boxed{2x - y + 4z = 12}$ or $\boxed{2x - y + 4z - 12 = 0}$
Verification: Point $(1, 2, 3)$ satisfies: $2(1) - 2 + 4(3) = 2 - 2 + 12 = 12$ ✓

Example 29: Plane Through Three Points

Problem: Find an equation of the plane through $P(1, 0, 2)$, $Q(2, 1, 1)$, and $R(0, 2, 3)$.

Solution:
Step 1: Find two vectors in the plane

$$\begin{aligned} \overrightarrow{PQ} &= \langle 2 - 1, 1 - 0, 1 - 2 \rangle = \langle 1, 1, -1 \rangle \\ \overrightarrow{PR} &= \langle 0 - 1, 2 - 0, 3 - 2 \rangle = \langle -1, 2, 1 \rangle \end{aligned}$$

Step 2: Find normal vector using cross product

$$\vec{n} = \overrightarrow{PQ} \times \overrightarrow{PR} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 1 & -1 \\ -1 & 2 & 1 \end{vmatrix}$$

$$\begin{aligned} &= \vec{i}(1 + 2) - \vec{j}(1 - 1) + \vec{k}(2 + 1) \\ &= 3\vec{i} - 0\vec{j} + 3\vec{k} \\ &= \langle 3, 0, 3 \rangle \end{aligned}$$

Step 3: Use point $P(1, 0, 2)$ and normal $\vec{n} = \langle 3, 0, 3 \rangle$

$$\begin{aligned} 3(x - 1) + 0(y - 0) + 3(z - 2) &= 0 \\ 3x - 3 + 3z - 6 &= 0 \\ 3x + 3z &= 9 \\ x + z &= 3 \end{aligned}$$

Answer: $\boxed{x + z = 3}$
Verification: Check that all three points satisfy the equation:

- $P(1, 0, 2)$: $1 + 2 = 3$ ✓
- $Q(2, 1, 1)$: $2 + 1 = 3$ ✓
- $R(0, 2, 3)$: $0 + 3 = 3$ ✓

Definition 6.26: Parallel and Perpendicular Planes

Two planes with normal vectors $\vec{n}_1$ and $\vec{n}_2$ are:

Parallel: if $\vec{n}_1 = k\vec{n}_2$ for some scalar k

$$\text{Plane}_1 \parallel \text{Plane}_2 \iff \vec{n}_1 \parallel \vec{n}_2$$

Perpendicular: if $\vec{n}_1 \cdot \vec{n}_2 = 0$

$$\text{Plane}_1 \perp \text{Plane}_2 \iff \vec{n}_1 \perp \vec{n}_2$$

Angle Between Planes:

The angle θ between two planes is the angle between their normal vectors:

$$\cos\theta = \frac{|\vec{n}_1 \cdot \vec{n}_2|}{|\vec{n}_1||\vec{n}_2|}$$

(We use absolute value to get the acute angle.)

Example 30: Parallel and Perpendicular Planes

Problem: Determine if the planes are parallel, perpendicular, or neither:

(a) $2x - y + 3z = 5$ and $4x - 2y + 6z = 1$

(b) $x + 2y - z = 3$ and $2x - y + 2z = 7$

Solution:

(a) Identify normal vectors:

$$\vec{n}_1 = \langle 2, -1, 3\rangle$$
$$\vec{n}_2 = \langle 4, -2, 6\rangle$$

Check if one is a multiple of the other:

$$\vec{n}_2 = 2\vec{n}_1 = 2\langle 2, -1, 3\rangle = \langle 4, -2, 6\rangle \quad \checkmark$$

Answer (a): $\boxed{\text{Parallel}}$

(b) Identify normal vectors:

$$\vec{n}_1 = \langle 1, 2, -1\rangle$$
$$\vec{n}_2 = \langle 2, -1, 2\rangle$$

Check dot product:

$$\vec{n}_1 \cdot \vec{n}_2 = (1)(2) + (2)(-1) + (-1)(2) = 2 - 2 - 2 = -2 \neq 0$$

Not perpendicular. Check if parallel:

$$\vec{n}_2 \neq k\vec{n}_1 \text{ for any scalar } k$$

Answer (b): $\boxed{\text{Neither parallel nor perpendicular}}$

Definition 6.27: Line-Plane Intersection

To find where a line intersects a plane:

Method:

1. Write line in parametric form: $x = x_0 + at$, $y = y_0 + bt$, $z = z_0 + ct$
2. Substitute into plane equation $Ax + By + Cz + D = 0$
3. Solve for parameter t
4. Substitute t back into parametric equations

Special Cases:

- If solution for t exists: line intersects plane at one point
- If $0 = 0$ (identity): line lies in the plane
- If $0 = k$ (contradiction): line is parallel to plane, no intersection

Example 31: Line-Plane Intersection

Problem: Find the point where the line $x = 1 + 2t$, $y = -1 + t$, $z = 3 - t$ intersects the plane $x + y + z = 5$.

Solution:

Step 1: Substitute parametric equations into plane equation

$$(1 + 2t) + (-1 + t) + (3 - t) = 5$$

Step 2: Solve for t

$$\begin{aligned} 1 + 2t - 1 + t + 3 - t &= 5 \\ 2t + 3 &= 5 \\ 2t &= 2 \\ t &= 1 \end{aligned}$$

Step 3: Find intersection point by substituting $t = 1$

$$\begin{aligned} x &= 1 + 2(1) = 3 \\ y &= -1 + 1 = 0 \\ z &= 3 - 1 = 2 \end{aligned}$$

Answer: $\boxed{(3, 0, 2)}$

Verification: $3 + 0 + 2 = 5$ ✓

Section 6.8 Summary: Planes in $\mathbb{R}^3$

Equation of a Plane:

$$\boxed{a(x - x_0) + b(y - y_0) + c(z - z_0) = 0}$$

or in general form:

$$\boxed{ax + by + cz + d = 0}$$

where $\vec{n} = \langle a, b, c \rangle$ is the normal vector.

Plane Through Three Points:

1. Find two vectors in the plane: $\overrightarrow{PQ}$ and $\overrightarrow{PR}$
2. Normal vector: $\vec{n} = \overrightarrow{PQ} \times \overrightarrow{PR}$
3. Use any of the three points with the normal vector

Parallel Planes: $\vec{n}_1 = k\vec{n}_2$
Perpendicular Planes: $\vec{n}_1 \cdot \vec{n}_2 = 0$
Line-Plane Intersection:

- Substitute line's parametric equations into plane equation
- Solve for parameter t
- Substitute back to find point

⚠ **Common Mistakes:**

- Using a direction vector instead of normal vector for plane
- Sign errors when expanding plane equation
- Not verifying that computed points satisfy the equation
- Confusing parallel lines with parallel planes (different tests!)

Key Insight: The normal vector is perpendicular to the plane, while direction vectors lie in the plane. For lines, we use direction vectors; for planes, we use normal vectors.

Cylindrical and Spherical Coordinates

In addition to Cartesian coordinates, cylindrical and spherical coordinate systems are useful for describing points and surfaces in three-dimensional space, especially when dealing with symmetry.

Definition 6.28: Cylindrical Coordinates

Cylindrical coordinates (r, θ, z) of a point P in $\mathbb{R}^3$ are defined as:

- r: distance from P to the z-axis (radial distance) $(r \geq 0)$
- θ: angle in the xy-plane from the positive x-axis $(0 \leq \theta < 2\pi)$
- z: height above the xy-plane (same as Cartesian z)

Conversion from Cylindrical to Cartesian:

$$\boxed{\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = z \end{cases}}$$

Conversion from Cartesian to Cylindrical:

$$\boxed{\begin{cases} r = \sqrt{x^2 + y^2} \\ \theta = \arctan\left(\frac{y}{x}\right) \\ z = z \end{cases}}$$

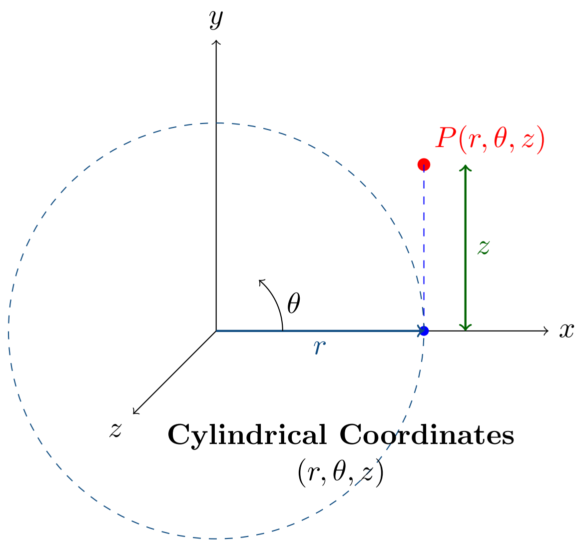
Interpretation:

Cylindrical coordinates extend polar coordinates (r, θ) from 2D to 3D by adding the z-coordinate.

When to Use:

Cylindrical coordinates are ideal for:

- Cylinders
- Surfaces with circular cross-sections
- Problems with rotational symmetry about the z-axis



Cylindrical Coordinates
(r, θ, z)

Example 32: Converting Cylindrical to Cartesian

Problem: Convert the cylindrical coordinates $(4, \frac{\pi}{3}, 5)$ to Cartesian coordinates.

Solution:
Given: $r = 4$, $\theta = \frac{\pi}{3}$, $z = 5$
Apply conversion formulas:

$$\begin{aligned}
x &= r\cos\theta = 4\cos\frac{\pi}{3} = 4 \cdot \frac{1}{2} = 2 \\
y &= r\sin\theta = 4\sin\frac{\pi}{3} = 4 \cdot \frac{\sqrt{3}}{2} = 2\sqrt{3} \\
z &= z = 5
\end{aligned}$$

Answer: $\boxed{(2, 2\sqrt{3}, 5)}$

Example 33: Converting Cartesian to Cylindrical

Problem: Convert the Cartesian point $(-3, 3, 7)$ to cylindrical coordinates.

Solution:
Given: $x = -3$, $y = 3$, $z = 7$
Step 1: Find r

$$r = \sqrt{x^2 + y^2} = \sqrt{(-3)^2 + 3^2} = \sqrt{9 + 9} = \sqrt{18} = 3\sqrt{2}$$

Step 2: Find θ
Since $x = -3 < 0$ and $y = 3 > 0$, the point is in the second quadrant.

$$\tan\theta = \frac{y}{x} = \frac{3}{-3} = -1$$

In the second quadrant: $\theta = \pi - \frac{\pi}{4} = \frac{3\pi}{4}$
Step 3: z remains the same

$$z = 7$$

Answer: $\boxed{\left(3\sqrt{2}, \frac{3\pi}{4}, 7\right)}$

Definition 6.29: Spherical Coordinates

Spherical coordinates (ρ, θ, ϕ) of a point P in $\mathbb{R}^3$ are defined as:

- ρ: distance from P to the origin ($\rho \geq 0$)
- θ: same angle as in cylindrical coordinates ($0 \leq \theta < 2\pi$)
- ϕ: angle from the positive z-axis down to $\overrightarrow{OP}$ ($0 \leq \phi \leq \pi$)

Conversion from Spherical to Cartesian:

$$\boxed{\begin{cases} x = \rho \sin\phi \cos\theta \\ y = \rho \sin\phi \sin\theta \\ z = \rho \cos\phi \end{cases}}$$

Conversion from Cartesian to Spherical:

$$\boxed{\begin{cases} \rho = \sqrt{x^2 + y^2 + z^2} \\ \theta = \arctan\left(\frac{y}{x}\right) \\ \phi = \arccos\left(\frac{z}{\sqrt{x^2+y^2+z^2}}\right) \end{cases}}$$

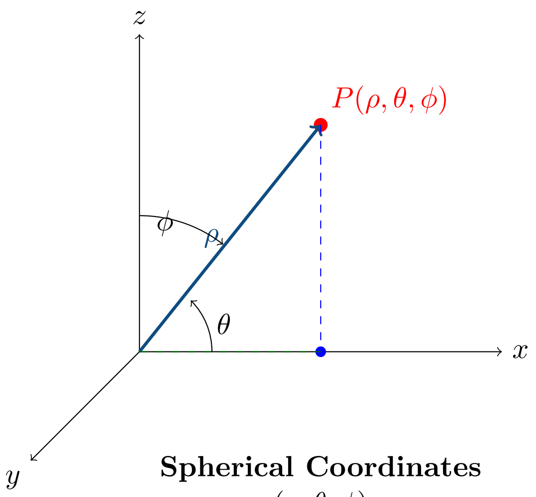
Relationship to Cylindrical:

$$\boxed{r = \rho \sin\phi, \quad z = \rho\cos\phi, \quad \rho = \sqrt{r^2 + z^2}}$$

When to Use:
Spherical coordinates are ideal for:

- Spheres
- Cones
- Problems with symmetry about the origin

Convention Note:
Some books use ϕ for the azimuthal angle and θ for the polar angle (opposite of our convention). Always check which convention is being used!



Spherical Coordinates
(ρ, θ, ϕ)

Example 34: Converting Spherical to Cartesian

Problem: Convert the spherical coordinates $(6, \frac{\pi}{4}, \frac{\pi}{3})$ to Cartesian coordinates.

Solution:
Given: $\rho = 6$, $\theta = \frac{\pi}{4}$, $\phi = \frac{\pi}{3}$
Apply conversion formulas:

$$\begin{aligned} x &= \rho \sin\phi \cos\theta = 6 \sin\frac{\pi}{3} \cos\frac{\pi}{4} \\ &= 6 \cdot \frac{\sqrt{3}}{2} \cdot \frac{\sqrt{2}}{2} = \frac{6\sqrt{6}}{4} = \frac{3\sqrt{6}}{2} \end{aligned}$$

$$\begin{aligned} y &= \rho \sin\phi \sin\theta = 6 \sin\frac{\pi}{3} \sin\frac{\pi}{4} \\ &= 6 \cdot \frac{\sqrt{3}}{2} \cdot \frac{\sqrt{2}}{2} = \frac{3\sqrt{6}}{2} \end{aligned}$$

$$z = \rho \cos\phi = 6 \cos\frac{\pi}{3} = 6 \cdot \frac{1}{2} = 3$$

Answer: $\boxed{\left(\frac{3\sqrt{6}}{2}, \frac{3\sqrt{6}}{2}, 3\right)}$

Example 35: Describing Surfaces

Problem: Describe the surfaces given by these equations:
(a) Cylindrical: $r = 3$
(b) Cylindrical: $\theta = \frac{\pi}{6}$
(c) Spherical: $\rho = 5$
(d) Spherical: $\phi = \frac{\pi}{4}$

Solution:

(a) $r = 3$ in cylindrical coordinates
This means the distance from the z-axis is constant (3 units).
Answer (a): $\boxed{\text{Circular cylinder of radius 3 about the } z\text{-axis}}$

(b) $\theta = \frac{\pi}{6}$ in cylindrical coordinates
The angle in the xy-plane is constant.
Answer (b): $\boxed{\text{Half-plane through the } z\text{-axis at angle } \dfrac{\pi}{6}}$

(c) $\rho = 5$ in spherical coordinates
Distance from origin is constant (5 units).
Answer (c): $\boxed{\text{Sphere of radius 5 centered at the origin}}$

(d) $\phi = \frac{\pi}{4}$ in spherical coordinates
Angle from positive z-axis is constant (45).
Answer (d): $\boxed{\text{Cone with vertex at origin, opening angle 45}}$

Section 6.9 Summary: Coordinate Systems

Cylindrical (r, θ, z)**:**

$$\boxed{\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = z \end{cases}} \qquad \boxed{\begin{cases} r = \sqrt{x^2 + y^2} \\ \theta = \arctan(y/x) \\ z = z \end{cases}}$$

Spherical (ρ, θ, ϕ)**:**

$$\boxed{\begin{cases} x = \rho\sin\phi\cos\theta \\ y = \rho\sin\phi\sin\theta \\ z = \rho\cos\phi \end{cases}} \qquad \boxed{\begin{cases} \rho = \sqrt{x^2 + y^2 + z^2} \\ \theta = \arctan(y/x) \\ \phi = \arccos(z/\rho) \end{cases}}$$

Common Surfaces:

Surface	Cylindrical	Spherical
Cylinder	$r = a$	$\rho\sin\phi = a$
Sphere	$r^2 + z^2 = a^2$	$\rho = a$
Cone	$z = kr$	$\phi = \text{const}$
Plane $z = c$	$z = c$	$\rho\cos\phi = c$

Common Mistakes:

- Confusing r (cylindrical) with ρ (spherical)
- Wrong quadrant for θ when converting from Cartesian
- Using degrees instead of radians
- Confusing angle conventions (check which is used!)

Curves in Space

This section explores special curves in three-dimensional space and introduces fundamental concepts like arc length and curvature.

Definition 6.30: Arc Length

If a smooth curve C is given by $\vec{r}(t) = \langle f(t), g(t), h(t)\rangle$ for $a \leq t \leq b$, then the **arc length** of C is:

$$\boxed{L = \int_a^b |\vec{r}'(t)|\, dt = \int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2 + [h'(t)]^2}\, dt}$$

Interpretation:
Arc length measures the actual distance traveled along the curve from $t = a$ to $t = b$.

Speed:
The speed at time t is $|\vec{r}'(t)|$.

Arc Length Function:
The arc length from $t = a$ to $t = s$ is:

$$s(t) = \int_a^t |\vec{r}'(u)|\, du$$

Example 36: Computing Arc Length

Problem: Find the length of the curve $\vec{r}(t) = \langle 2t, t^2, \frac{1}{3}t^3\rangle$ from $t = 0$ to $t = 3$.

Solution:
Step 1: Find $\vec{r}'(t)$

$$\vec{r}'(t) = \langle 2, 2t, t^2\rangle$$

Step 2: Find $|\vec{r}'(t)|$

$$\begin{aligned} |\vec{r}'(t)| &= \sqrt{2^2 + (2t)^2 + (t^2)^2} \\ &= \sqrt{4 + 4t^2 + t^4} \\ &= \sqrt{(t^2 + 2)^2} \\ &= t^2 + 2 \end{aligned}$$

Step 3: Integrate

$$\begin{aligned} L &= \int_0^3 (t^2 + 2)\, dt \\ &= \left[\frac{t^3}{3} + 2t\right]_0^3 \\ &= \frac{27}{3} + 6 - 0 \\ &= 9 + 6 \\ &= 15 \end{aligned}$$

Answer: $\boxed{L = 15 \text{ units}}$

Definition 6.31: Curvature

The **curvature** κ of a curve at a point measures how sharply the curve bends at that point.

Formula 1 (using arc length parameter):

$$\boxed{\kappa = \left|\frac{d\vec{T}}{ds}\right|}$$

where $\vec{T}$ is the unit tangent vector and s is arc length.

Formula 2 (using parameter t):

$$\boxed{\kappa(t) = \frac{|\vec{T}'(t)|}{|\vec{r}'(t)|}}$$

Formula 3 (practical formula):

$$\boxed{\kappa(t) = \frac{|\vec{r}'(t) \times \vec{r}''(t)|}{|\vec{r}'(t)|^3}}$$

Interpretation:

- $\kappa = 0$: curve is a straight line at that point
- Large κ: curve bends sharply
- Small κ: curve bends gently

Radius of Curvature:

$$\rho = \frac{1}{\kappa}$$

This is the radius of the "best-fit" circle at that point.

Example 37: Computing Curvature

Problem: Find the curvature of the helix $\vec{r}(t) = \langle \cos t, \sin t, t\rangle$.

Solution:

Step 1: Find $\vec{r}'(t)$ and $\vec{r}''(t)$

$$\vec{r}'(t) = \langle -\sin t, \cos t, 1\rangle$$
$$\vec{r}''(t) = \langle -\cos t, -\sin t, 0\rangle$$

Step 2: Compute cross product $\vec{r}'(t) \times \vec{r}''(t)$

$$\vec{r}'(t) \times \vec{r}''(t) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -\sin t & \cos t & 1 \\ -\cos t & -\sin t & 0 \end{vmatrix}$$

$$= \vec{i}(0+\sin t) - \vec{j}(0+\cos t) + \vec{k}(\sin^2 t + \cos^2 t)$$
$$= \langle \sin t, -\cos t, 1\rangle$$

Step 3: Find magnitudes

$$|\vec{r}'(t) \times \vec{r}''(t)| = \sqrt{\sin^2 t + \cos^2 t + 1} = \sqrt{2}$$
$$|\vec{r}'(t)| = \sqrt{\sin^2 t + \cos^2 t + 1} = \sqrt{2}$$

Step 4: Apply curvature formula

$$\kappa(t) = \frac{|\vec{r}'(t) \times \vec{r}''(t)|}{|\vec{r}'(t)|^3} = \frac{\sqrt{2}}{(\sqrt{2})^3} = \frac{\sqrt{2}}{2\sqrt{2}} = \frac{1}{2}$$

Answer: $\boxed{\kappa = \frac{1}{2}}$ (constant curvature)

Interpretation: The helix has constant curvature $\frac{1}{2}$ at every point.

Definition 6.32: Special Space Curves

1. Circular Helix:

$$\vec{r}(t) = \langle a\cos t, a\sin t, bt\rangle$$

Properties:

- Projects to circle of radius a in xy-plane
- Rises at constant rate b (pitch)
- Constant curvature: $\kappa = \frac{a}{a^2+b^2}$

2. Twisted Cubic (Space Curve):

$$\vec{r}(t) = \langle t, t^2, t^3\rangle$$

3. Astroid (when projected to a plane):
In the xy-plane:

$$\vec{r}(t) = \langle a\cos^3 t, a\sin^3 t\rangle$$

This is a star-shaped curve with four cusps.
Arc length of astroid:

$$L = 6a \quad \text{(one complete loop)}$$

Properties:
The astroid is the path traced by a point on a small circle rolling inside a larger circle (with radius ratio 4:1).

Example 38: Arc Length of Astroid

Problem: Find the arc length of one loop of the astroid $\vec{r}(t) = \langle \cos^3 t, \sin^3 t \rangle$ for $0 \leq t \leq 2\pi$.

Solution:
Step 1: Find $\vec{r}'(t)$

$$\vec{r}'(t) = \langle -3\cos^2 t \sin t, 3\sin^2 t \cos t \rangle$$

Step 2: Find $|\vec{r}'(t)|$

$$\begin{aligned}
|\vec{r}'(t)| &= \sqrt{9\cos^4 t \sin^2 t + 9\sin^4 t \cos^2 t} \\
&= 3\sqrt{\cos^4 t \sin^2 t + \sin^4 t \cos^2 t} \\
&= 3\sqrt{\cos^2 t \sin^2 t(\cos^2 t + \sin^2 t)} \\
&= 3|\cos t \sin t| \\
&= 3|\sin t \cos t|
\end{aligned}$$

Step 3: Set up integral (using symmetry)
Due to symmetry, we can compute for one quarter ($0 \to \frac{\pi}{2}$) and multiply by 4:

$$\begin{aligned}
L &= 4\int_0^{\pi/2} 3\sin t \cos t \, dt \\
&= 12\int_0^{\pi/2} \sin t \cos t \, dt \\
&= 12 \cdot \frac{1}{2}\int_0^{\pi/2} \sin(2t) \, dt \\
&= 6\left[-\frac{1}{2}\cos(2t)\right]_0^{\pi/2} \\
&= 6 \cdot \frac{1}{2}[1 - (-1)] \\
&= 3 \cdot 2 \\
&= 6
\end{aligned}$$

Answer: $\boxed{L = 6 \text{ units}}$ (for $a = 1$)
General: For astroid with parameter a: $L = 6a$

Section 6.10 Summary: Curves in Space

Arc Length:

$$\boxed{L = \int_a^b |\vec{r}'(t)| \, dt}$$

Curvature:

$$\boxed{\kappa(t) = \frac{|\vec{r}'(t) \times \vec{r}''(t)|}{|\vec{r}'(t)|^3}}$$

Key Formulas:

- Speed: $|\vec{r}'(t)|$
- Unit tangent: $\vec{T}(t) = \frac{\vec{r}'(t)}{|\vec{r}'(t)|}$
- Radius of curvature: $\rho = \frac{1}{\kappa}$

Special Curves:

- Helix: $\vec{r}(t) = \langle a\cos t, a\sin t, bt\rangle$ (constant curvature)
- Astroid: $\vec{r}(t) = \langle a\cos^3 t, a\sin^3 t\rangle$ (arc length $6a$)
- Twisted cubic: $\vec{r}(t) = \langle t, t^2, t^3\rangle$

⚠ **Common Mistakes:**

- Forgetting to take magnitude of $\vec{r}'(t)$ in arc length
- Sign errors in cross product for curvature
- Not using symmetry to simplify arc length integrals
- Confusing curvature with radius of curvature

Key Insight: Curvature measures how sharply a curve bends; arc length measures total distance along the curve.

Practice Exercises

Problem 1: Magnitude and Direction

(a) Find the magnitude of $\vec{v} = \langle 2, -3, 6 \rangle$.
(b) Find a unit vector in the direction of $\vec{u} = \langle 4, -4, 2 \rangle$.
(c) Find the direction cosines of $\vec{w} = \langle 3, 4, 0 \rangle$.
(d) Find a vector of length 10 parallel to $\vec{v} = \langle 1, 2, 2 \rangle$.
(e) Find the distance between points $P(2, -1, 3)$ and $Q(5, 3, -1)$.
(f) Find the midpoint between $A(1, 4, -2)$ and $B(3, -2, 6)$.

Problem 2: Vectors Between Points

(a) Find the vector from $P(1, 2, -1)$ to $Q(3, 0, 4)$.
(b) Given points $A(2, 1, 3)$, $B(4, 0, 1)$, and $C(6, -1, -1)$, show that these three points are collinear.
(c) Find all points on the z-axis that are 5 units from point $P(3, 4, 0)$.
(d) Find the point P such that $\overrightarrow{OP}$ is parallel to $\vec{v} = \langle 2, -1, 3 \rangle$ and $|\overrightarrow{OP}| = 7$.

Problem 3: Special Vectors

(a) Express $\vec{v} = \langle 3, -2, 5 \rangle$ in terms of the standard unit vectors $\vec{i}$, $\vec{j}$, $\vec{k}$.
(b) Find scalars a and b such that $\langle 6, -2, 4 \rangle = a\langle 2, 1, -1 \rangle + b\langle 1, -2, 3 \rangle$.
(c) Determine if vectors $\vec{u} = \langle 2, -3, 1 \rangle$ and $\vec{v} = \langle -4, 6, -2 \rangle$ are parallel.
(d) Find two unit vectors perpendicular to $\vec{v} = \langle 1, 1, 0 \rangle$.

Problem 4: Vector Arithmetic

Given $\vec{u} = \langle 3, -1, 2 \rangle$, $\vec{v} = \langle 1, 4, -2 \rangle$, and $\vec{w} = \langle -2, 0, 3 \rangle$:
(a) Find $2\vec{u} - 3\vec{v}$
(b) Find $|\vec{u} + \vec{v}|$
(c) Find a vector $\vec{x}$ such that $2\vec{x} + \vec{u} = 3\vec{v} - \vec{w}$
(d) Verify that $|\vec{u} + \vec{v}| \leq |\vec{u}| + |\vec{v}|$ (triangle inequality)
(e) Find scalars a and b such that $\vec{w} = a\vec{u} + b\vec{v}$ (if possible)

Problem 5: Linear Combinations

(a) Express $\langle 7, -3, 10 \rangle$ as a linear combination of $\vec{u} = \langle 1, 2, 1 \rangle$ and $\vec{v} = \langle 2, -1, 3 \rangle$.
(b) Determine if $\vec{w} = \langle 5, 1, 4 \rangle$ can be written as a linear combination of $\vec{u} = \langle 1, 0, 1 \rangle$ and $\vec{v} = \langle 0, 1, 1 \rangle$.
(c) Find the position vector $\vec{r}$ such that $\vec{r} = 2\vec{a} - 3\vec{b}$ where $\vec{a}$ and $\vec{b}$ are position vectors of points $A(1, 2, 3)$ and $B(-1, 1, 2)$.

Problem 6: Dot Product Calculations

(a) Find $\vec{u} \cdot \vec{v}$ where $\vec{u} = \langle 2, -3, 4 \rangle$ and $\vec{v} = \langle 1, 2, -1 \rangle$.
(b) Find the angle between $\vec{u} = \langle 1, 2, 2 \rangle$ and $\vec{v} = \langle 2, 1, -2 \rangle$.
(c) Determine if $\vec{a} = \langle 3, -1, 2 \rangle$ and $\vec{b} = \langle 1, 5, -2 \rangle$ are orthogonal.

(d) Find k such that $\vec{u} = \langle 2, k, -3\rangle$ and $\vec{v} = \langle k, 1, 2\rangle$ are perpendicular.
(e) Verify that $\vec{u} \cdot (\vec{v} + \vec{w}) = \vec{u} \cdot \vec{v} + \vec{u} \cdot \vec{w}$ for specific vectors.

Problem 7: Orthogonal Projections

(a) Find the projection of $\vec{u} = \langle 3, 4, -1\rangle$ onto $\vec{v} = \langle 2, 1, 2\rangle$.
(b) Find the scalar component of $\vec{a} = \langle 1, 2, 3\rangle$ in the direction of $\vec{b} = \langle 2, -1, 2\rangle$.
(c) Decompose $\vec{u} = \langle 4, 1, 5\rangle$ into components parallel and perpendicular to $\vec{v} = \langle 1, 2, 1\rangle$.
(d) Find the work done by force $\vec{F} = \langle 3, -2, 4\rangle$ N moving an object from $P(1, 0, 2)$ to $Q(4, 3, 5)$.
(e) A force of 50 N acts at angle 30 to the direction of motion. If the object moves 10 m, find the work done.

Problem 8: Applications of Dot Product

(a) Show that the triangle with vertices $A(1, 0, 1)$, $B(2, 3, 1)$, and $C(3, 2, 4)$ is a right triangle.
(b) Find the angle at vertex B in triangle ABC where $A(1, 1, 2)$, $B(3, -1, 4)$, $C(0, 2, 1)$.
(c) Determine if vectors $\vec{u} = \langle 1, 2, -1\rangle$, $\vec{v} = \langle 2, 1, 3\rangle$, and $\vec{w} = \langle 3, -1, 1\rangle$ form an orthogonal set.
(d) Find a vector orthogonal to both $\vec{u} = \langle 1, 1, 0\rangle$ and $\vec{v} = \langle 0, 1, 1\rangle$ using projections.

Problem 9: Cross Product Calculations

(a) Find $\vec{u} \times \vec{v}$ where $\vec{u} = \langle 1, 3, -2\rangle$ and $\vec{v} = \langle 2, -1, 4\rangle$.
(b) Verify that $\vec{u} \times \vec{v}$ is perpendicular to both $\vec{u}$ and $\vec{v}$ from part (a).
(c) Find $|\vec{u} \times \vec{v}|$ where $\vec{u} = \langle 2, 0, -1\rangle$ and $\vec{v} = \langle 1, 3, 2\rangle$.
(d) Verify that $\vec{u} \times \vec{v} = -(\vec{v} \times \vec{u})$ for specific vectors.
(e) Find a unit vector perpendicular to both $\vec{a} = \langle 1, 2, 1\rangle$ and $\vec{b} = \langle 2, 1, -1\rangle$.

Problem 10: Geometric Applications

(a) Find the area of the parallelogram with adjacent sides $\vec{u} = \langle 2, 1, 0\rangle$ and $\vec{v} = \langle 1, 3, 2\rangle$.
(b) Find the area of triangle ABC where $A(1, 0, 2)$, $B(3, 1, 0)$, and $C(2, 4, 1)$.
(c) Find the volume of the parallelepiped determined by vectors $\vec{u} = \langle 1, 2, 3\rangle$, $\vec{v} = \langle 2, 1, 0\rangle$, and $\vec{w} = \langle 0, 1, 2\rangle$.
(d) Determine if vectors $\vec{a} = \langle 1, 2, 1\rangle$, $\vec{b} = \langle 2, 1, 3\rangle$, and $\vec{c} = \langle 3, 3, 4\rangle$ are coplanar.
(e) Find the distance from point $P(1, 0, 2)$ to the line through $Q(0, 1, 1)$ with direction $\vec{v} = \langle 1, 1, -1\rangle$.

Problem 11: Mixed Products

(a) Compute $\vec{u} \cdot (\vec{v} \times \vec{w})$ where $\vec{u} = \langle 1, 2, 0\rangle$, $\vec{v} = \langle 0, 1, 3\rangle$, $\vec{w} = \langle 2, 0, 1\rangle$.
(b) Verify the identity $(\vec{u} \times \vec{v}) \cdot \vec{w} = \vec{u} \cdot (\vec{v} \times \vec{w})$ for specific vectors.
(c) Find the volume of tetrahedron with vertices $O(0, 0, 0)$, $A(2, 1, 0)$, $B(1, 3, 1)$, $C(0, 1, 4)$.

Hint: Volume $= \frac{1}{6}|\vec{OA} \cdot (\vec{OB} \times \vec{OC})|$
(d) Show that $\vec{u} \times (\vec{v} \times \vec{w}) \neq (\vec{u} \times \vec{v}) \times \vec{w}$ using specific vectors.

Problem 12: Vector Function Basics

(a) Find $\vec{r}'(t)$ and $\vec{r}''(t)$ for $\vec{r}(t) = \langle t^2, e^t, \sin t\rangle$.
(b) Find $\lim_{t\to 0} \vec{r}(t)$ where $\vec{r}(t) = \left\langle \frac{\sin t}{t}, t^2 + 1, e^{2t}\right\rangle$.
(c) Find the unit tangent vector $\vec{T}(t)$ for $\vec{r}(t) = \langle \cos t, \sin t, t\rangle$ at $t = \frac{\pi}{4}$.
(d) Evaluate $\displaystyle\int_0^1 \langle 2t, 3t^2, e^t\rangle\, dt$.
(e) Find $\vec{r}(t)$ if $\vec{r}'(t) = \langle 6t, 2, 3t^2\rangle$ and $\vec{r}(0) = \langle 1, 2, -1\rangle$.

Problem 13: Calculus of Vector Functions

(a) Find $\dfrac{d}{dt}[\vec{u}(t) \cdot \vec{v}(t)]$ where $\vec{u}(t) = \langle t, t^2, t^3\rangle$ and $\vec{v}(t) = \langle \cos t, \sin t, t\rangle$.
(b) Find $\dfrac{d}{dt}[\vec{u}(t) \times \vec{v}(t)]$ for $\vec{u}(t) = \langle t, 1, t^2\rangle$ and $\vec{v}(t) = \langle 2t, t, 1\rangle$.
(c) If $|\vec{r}(t)|$ is constant, show that $\vec{r}(t) \perp \vec{r}'(t)$.
(d) Find $\displaystyle\int \langle te^{t^2}, \cos(2t), t^{-1}\rangle\, dt$ for $t > 0$.
(e) Solve the initial value problem: $\vec{r}''(t) = \langle 0, 0, -g\rangle$, $\vec{r}'(0) = \langle v_0 \cos\alpha, v_0 \sin\alpha, 0\rangle$, $\vec{r}(0) = \vec{0}$.

Problem 14: Space Curves

(a) Describe the curve given by $\vec{r}(t) = \langle 2\cos t, 2\sin t, 3\rangle$ for $0 \leq t \leq 2\pi$.
(b) Find a vector function for the curve of intersection of the cylinder $x^2 + y^2 = 4$ and plane $z = x + 2$.
(c) Sketch and identify the curve $\vec{r}(t) = \langle t, t^2, 0\rangle$.
(d) Find the point where the curves $\vec{r}_1(t) = \langle t, 1-t, 3+t^2\rangle$ and $\vec{r}_2(s) = \langle 3-s, s-2, s^2\rangle$ intersect (if they do).

Problem 15: Equations of Lines

(a) Find parametric equations for the line through $P(2, 1, 3)$ parallel to $\vec{v} = \langle 1, -2, 4\rangle$.
(b) Find symmetric equations for the line through $P(1, 0, -1)$ and $Q(3, 2, 4)$.
(c) Find parametric equations for the line through $(3, -1, 2)$ perpendicular to the plane $2x - y + 3z = 5$.
(d) Convert the symmetric equations $\frac{x-1}{2} = \frac{y+3}{-1} = \frac{z}{4}$ to parametric form.
(e) Find vector equation of the line passing through $(1, 2, 3)$ parallel to the y-axis.

Problem 16: Line Relationships

(a) Determine if the lines are parallel, perpendicular, or neither:

$$L_1: \quad x = 2 + t,\ y = 1 - 2t,\ z = 3t$$

$$L_2: \quad x = 1 - 2s,\ y = 3 + 4s,\ z = 2 - 6s$$

(b) Find the acute angle between the lines:

$$L_1: \quad \frac{x-1}{2} = \frac{y}{3} = \frac{z+1}{1}$$

$$L_2: \quad \frac{x+2}{1} = \frac{y-1}{2} = \frac{z}{-1}$$

(c) Show that the lines $x = 1 + t$, $y = 2 - t$, $z = 3 + 2t$ and $x = 3 - s$, $y = 1 + s$, $z = -1 + 4s$ are skew.

(d) Find the distance between parallel lines if they exist in part (a).

Problem 17: Line Intersections

(a) Determine if the lines intersect. If so, find the point of intersection:

$$L_1: \quad x = 1 + 2t,\ y = 3 - t,\ z = 4 + t$$

$$L_2: \quad x = 5 - s,\ y = 1 + 2s,\ z = 7 - 2s$$

(b) Find equations of the line through $(2, 1, 0)$ that intersects and is perpendicular to the line $x = 1 + t$, $y = 2t$, $z = 3 - t$.

(c) Find the shortest distance from point $P(3, 1, 2)$ to the line $\vec{r}(t) = \langle 1, 2, 1\rangle + t\langle 2, 1, -1\rangle$.

Problem 18: Equations of Planes

(a) Find an equation of the plane through $P(2, 1, 3)$ with normal vector $\vec{n} = \langle 1, -2, 3\rangle$.

(b) Find an equation of the plane through points $P(1, 0, 2)$, $Q(2, 1, 1)$, and $R(0, 2, 3)$.

(c) Find an equation of the plane containing the line $x = 1 + t$, $y = 2 - t$, $z = 3 + 2t$ and the point $(0, 1, 4)$.

(d) Find an equation of the plane through $(1, 2, 3)$ parallel to the plane $2x - y + 3z = 5$.

(e) Find the plane through $(2, 1, 0)$ perpendicular to the line of intersection of planes $x + y + z = 1$ and $x - y + 2z = 3$.

Problem 19: Plane Relationships

(a) Determine if the planes are parallel, perpendicular, or neither:

$$P_1: \quad 2x - y + 3z = 5$$

$$P_2: \quad 4x - 2y + 6z = 1$$

(b) Find the acute angle between the planes $x + y + z = 1$ and $x - 2y + 3z = 4$.

(c) Find the distance from point $P(1, 2, 3)$ to the plane $2x - 2y + z = 5$.

(d) Find an equation of the plane that passes through the point $(1, 0, 2)$ and is perpendicular to both planes $x + y + z = 1$ and $x - y + z = 0$.

Problem 20: Special Planes

(a) Find the equation of the plane through the origin parallel to the plane containing points $(1, 0, 0)$, $(0, 1, 0)$, and $(0, 0, 1)$.

(b) Find the equation of the plane that contains the lines:

$$L_1 : \quad x = 1 + t,\ y = t,\ z = 2 - t$$

$$L_2 : \quad x = 1 + 2s,\ y = s,\ z = 2 + s$$

(c) Find the equation of the plane equidistant from points $P(2, 1, 0)$ and $Q(0, 3, 2)$.

(d) Find the trace of the plane $2x - 3y + z = 6$ in each coordinate plane.

Problem 21: Line-Plane Intersections

(a) Find the point where the line $x = 1 + 2t$, $y = 3 - t$, $z = 4 + 3t$ intersects the plane $x + y + z = 10$.

(b) Determine if the line $\vec{r}(t) = \langle 2, 1, 3\rangle + t\langle 1, -1, 2\rangle$ is parallel to, lies in, or intersects the plane $x - y + z = 4$.

(c) Find the angle between the line $x = t$, $y = 1 + t$, $z = 2 - t$ and the plane $2x + y - z = 5$.

(d) Find symmetric equations for the line through $(3, 1, 2)$ parallel to the line of intersection of planes $x + y - z = 2$ and $2x - y + 3z = 1$.

Problem 22: Plane-Plane Intersections

(a) Find parametric equations for the line of intersection of planes $x + y - z = 1$ and $2x - y + 3z = 4$.

(b) Determine if the three planes intersect at a single point:

$$x + y + z = 6, \quad 2x - y + z = 3, \quad x + 2y - z = 0$$

(c) Find the distance between the parallel planes $2x - y + 3z = 5$ and $2x - y + 3z = 10$.

(d) Find the equation of the plane through the line of intersection of $x + y + z = 1$ and $x - y + 2z = 3$ that passes through the origin.

Problem 23: Complex Intersections

(a) Find the point of intersection of the line $\frac{x-1}{2} = \frac{y+1}{3} = \frac{z-2}{-1}$ and the plane $3x - 2y + z = 10$.

(b) Find the line of intersection of the planes $x + 2y - z = 3$ and $2x - y + 3z = 1$, and find where this line intersects the xy-plane.

(c) Determine if the line $x = 1 + t$, $y = 2t$, $z = 3 - t$ intersects both planes $x + y + z = 6$ and $x - y = 1$.

(d) Find the distance from the line $\vec{r}(t) = \langle 1, 0, 2\rangle + t\langle 1, 1, -1\rangle$ to the plane $2x - y + z = 4$.

Problem 24: Cylindrical Coordinates

(a) Convert the point $(3, 3\sqrt{3}, 5)$ from Cartesian to cylindrical coordinates.

(b) Convert the point $(4, \frac{2\pi}{3}, -2)$ from cylindrical to Cartesian coordinates.

(c) Describe the surface $r = 4$ in cylindrical coordinates.

(d) Convert the equation $x^2 + y^2 = 9$ to cylindrical coordinates.

(e) Find the cylindrical coordinates of the point with Cartesian coordinates $(-2, 2, 1)$.

Problem 25: Spherical Coordinates

(a) Convert the point $(2, 2\sqrt{3}, 4)$ from Cartesian to spherical coordinates.

(b) Convert the point $(6, \frac{\pi}{4}, \frac{\pi}{3})$ from spherical to Cartesian coordinates.

(c) Describe the surface $\rho = 3$ in spherical coordinates.

(d) Convert the equation $x^2 + y^2 + z^2 = 16$ to spherical coordinates.

(e) Describe the surface $\phi = \frac{\pi}{3}$ in spherical coordinates.

Problem 26: Mixed Coordinate Systems

(a) Convert the cylindrical equation $r = 2\sin\theta$ to Cartesian form.

(b) Convert the spherical equation $\rho = 4\cos\phi$ to cylindrical and then to Cartesian form.

(c) A point has cylindrical coordinates $(2, \frac{\pi}{6}, 3)$. Find its spherical coordinates.

(d) Describe the cone $z = \sqrt{x^2 + y^2}$ in both cylindrical and spherical coordinates.

(e) Find the volume of the region bounded by $z = \sqrt{x^2 + y^2}$ and $z = 2$ using appropriate coordinates.

Problem 27: Arc Length

(a) Find the length of the curve $\vec{r}(t) = \langle 2t, t^2, \frac{2}{3}t^3 \rangle$ from $t = 0$ to $t = 3$.

(b) Find the arc length function for $\vec{r}(t) = \langle e^t \cos t, e^t \sin t, e^t \rangle$ starting at $t = 0$.

(c) Find the length of one turn of the helix $\vec{r}(t) = \langle 3\cos t, 3\sin t, 4t \rangle$ from $t = 0$ to $t = 2\pi$.

(d) Parametrize the line segment from $(1, 0, 2)$ to $(3, 4, -1)$ using arc length as parameter.

Problem 28: Curvature

(a) Find the curvature of $\vec{r}(t) = \langle t, t^2, t^3 \rangle$ at $t = 1$.

(b) Find the curvature of the circle $\vec{r}(t) = \langle 5\cos t, 5\sin t, 0 \rangle$.

(c) Find the point on the curve $\vec{r}(t) = \langle t, t^2, 0 \rangle$ where the curvature is maximum.

(d) Show that the curvature of a straight line is zero.

(e) Find the radius of curvature of $\vec{r}(t) = \langle \cos t, \sin t, t \rangle$ at any point.

Problem 29: Special Curves

(a) Find the arc length of the astroid $x = a\cos^3 t$, $y = a\sin^3 t$ for one complete loop ($0 \le t \le 2\pi$).

(b) Verify that the helix $\vec{r}(t) = \langle a\cos t, a\sin t, bt\rangle$ has constant curvature $\kappa = \frac{a}{a^2+b^2}$.

(c) For the twisted cubic $\vec{r}(t) = \langle t, t^2, t^3\rangle$, find where the curvature is minimum.

(d) Find the curvature of the curve of intersection of the cylinder $x^2 + y^2 = 4$ and the plane $z = x$.

Problem 30: Applications

(a) A particle moves along the curve $\vec{r}(t) = \langle t^2, 2t, \ln t\rangle$ for $t > 0$. Find its velocity, speed, and acceleration at $t = 1$.

(b) Find the tangential and normal components of acceleration for a particle with position $\vec{r}(t) = \langle \cos t, \sin t, t\rangle$.

(c) A particle moves with constant speed along the curve $\vec{r}(t) = \langle e^t\cos t, e^t\sin t, e^t\rangle$. Is this possible? Explain.

(d) Find the osculating circle (circle of curvature) for $\vec{r}(t) = \langle t, t^2, 0\rangle$ at $t = 0$.

Chapter 6 Practice Problems – Summary

Coverage by Topic:

Topic	Problems	Exam Frequency
Dot Product & Projections	6–8	**Frequent**
Cross Product	9–11	**Frequent**
Lines in $\mathbb{R}^3$	15–17	High
Planes in $\mathbb{R}^3$	18–20	High
Intersections	21–23	High
Vector Functions	12–14	Medium-High
Coordinate Systems	24–26	Medium
Curvature	27–30	Medium
Vector Basics	1–3	Medium
Vector Operations	4–5	Medium