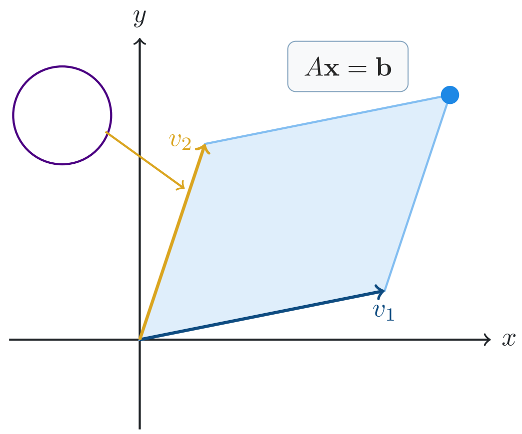
Linear Algebra

ADP / BS (IT - CS - SE)

A Comprehensive Course Companion



A journey through matrices, vector spaces, linear mappings, and the structure of inner product spaces.

Sohaib Hasan (HOD Mathematics) • Horizon College Chakwal

sohaibhassan199@gmail.com • +92-3335904314 •

∞ META

@plotlab01 /plotlab1

Contents

Chapter 1

Matrics & Determinants

Matrices

Matrix

A **matrix** is a rectangular array of numbers arranged in rows and columns, enclosed in square brackets. A matrix having m rows and n columns is called an $m \times n$ **matrix**. The number in row i and column j is called the (i, j)**-entry** and is denoted a_{ij}.

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

We write $A = [a_{ij}]$ for short and say that A has **order** $m \times n$. Two matrices are **equal** if and only if they have the same order and every corresponding entry is equal.

Example 1.1 — Identifying Size and Entries

Let $A = \begin{bmatrix} 5 & -3 & 0 \\ 2 & 1 & 7 \end{bmatrix}$. Find the size of A and identify a_{12}, a_{21} and a_{23}.

A has 2 rows and 3 columns, so it is a 2×3 matrix. Reading entries by their row-column address:

$$a_{12} = -3, \qquad a_{21} = 2, \qquad a_{23} = 7.$$

Example 1.2 — Reading Entries of a Matrix

Let $B = \begin{bmatrix} 4 & 0 & -1 \\ 7 & 3 & 2 \\ -5 & 8 & 1 \end{bmatrix}$. State the size of B and find b_{31} and b_{13}.

B has 3 rows and 3 columns, so it is a 3×3 matrix.

$$b_{31} = -5, \qquad b_{13} = -1.$$

Example 1.3 — Equality of Matrices

Find x and y if $\begin{bmatrix} x+y & 2 \\ 3 & x-y \end{bmatrix} = \begin{bmatrix} 5 & 2 \\ 3 & 1 \end{bmatrix}$.

Equal matrices must have every corresponding entry equal, so:

$$x + y = 5 \qquad \text{and} \qquad x - y = 1.$$

Adding: $2x = 6 \Rightarrow x = 3$. Substituting: $3 + y = 5 \Rightarrow y = 2$.

Special Types of Matrices

- **Row Matrix:** A matrix having exactly one row.
- **Column Matrix:** A matrix having exactly one column.
- **Square Matrix:** A matrix in which the number of rows equals the number of columns, i.e. $m = n$.
- **Zero Matrix (0):** A matrix in which every entry is zero.
- **Identity Matrix (I_n):** The $n \times n$ square matrix with 1s on the main diagonal and 0s elsewhere:

$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \qquad I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Example 1.4 — Identifying Types of Matrices

Identify the type of each matrix.

$$A = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}, \quad B = \begin{bmatrix} 4 \\ -1 \\ 0 \end{bmatrix}, \quad C = \begin{bmatrix} 2 & 5 \\ 3 & -1 \end{bmatrix}, \quad D = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

- A is a 1×3 **row matrix**.
- B is a 3×1 **column matrix**.
- C is a 2×2 **square matrix**.
- D is a 2×2 **zero matrix**.

Example 1.5 -Past Paper 2024 Q1(i) 2025 Q1(ix)

If B and C are both inverses of a matrix A, prove that $B = C$.

Since B is an inverse of A: $BA = I$.
Since C is an inverse of A: $AC = I$.
Multiply both sides of $BA = I$ on the right by C:

$$(BA)C = IC \Rightarrow B(AC) = C \Rightarrow BI = C \Rightarrow B = C.$$

Hence any matrix has **at most one** inverse.

Example 1.6 — Identity Matrix Check

For what value of k does $\begin{bmatrix} k+1 & 0 \\ 0 & 2k-3 \end{bmatrix} = I_2$?

We need both diagonal entries to equal 1:

$$k + 1 = 1 \Rightarrow k = 0, \qquad 2k - 3 = 1 \Rightarrow k = 2.$$

The two conditions give different values of k, so **no** single value of k makes this matrix equal to I_2.

Algebra of Matrices

Matrix Addition

Let $A = [a_{ij}]$ and $B = [b_{ij}]$ be two $m \times n$ matrices. Their **sum** $A + B$ is the $m \times n$ matrix obtained by adding corresponding entries:

$$(A + B)_{ij} = a_{ij} + b_{ij}.$$

Matrix addition is defined **only** when A and B have the same order. The **difference** is defined as $A - B = [a_{ij} - b_{ij}]$.

Example 1.7 — Matrix Addition and Subtraction

Let $A = \begin{bmatrix} 1 & -2 \\ 3 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 4 & 1 \\ -1 & 5 \end{bmatrix}$. Find $A + B$ and $A - B$.

$$A + B = \begin{bmatrix} 1+4 & -2+1 \\ 3-1 & 0+5 \end{bmatrix} = \begin{bmatrix} 5 & -1 \\ 2 & 5 \end{bmatrix}.$$

$$A - B = \begin{bmatrix} 1-4 & -2-1 \\ 3+1 & 0-5 \end{bmatrix} = \begin{bmatrix} -3 & -3 \\ 4 & -5 \end{bmatrix}.$$

Scalar Multiplication

Let $A = [a_{ij}]$ be an $m \times n$ matrix and let k be a real number (called a **scalar**). The **scalar multiple** kA is the $m \times n$ matrix obtained by multiplying every entry of A by k:

$$(kA)_{ij} = k \cdot a_{ij}.$$

The **negative** of A is defined as $-A = (-1)A$.

Example 1.8 — Scalar Multiplication

Let $A = \begin{bmatrix} 2 & -1 \\ 0 & 4 \end{bmatrix}$. Find $3A$ and $-2A$.

$$3A = \begin{bmatrix} 6 & -3 \\ 0 & 12 \end{bmatrix}, \qquad -2A = \begin{bmatrix} -4 & 2 \\ 0 & -8 \end{bmatrix}.$$

Example 1.9 — Combined Addition and Scalar Multiplication

Let $A = \begin{bmatrix} 1 & -2 \\ 3 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 4 & 1 \\ -1 & 5 \end{bmatrix}$. Compute $2A - 3B$.

$$2A = \begin{bmatrix} 2 & -4 \\ 6 & 0 \end{bmatrix}, \qquad 3B = \begin{bmatrix} 12 & 3 \\ -3 & 15 \end{bmatrix}.$$

$$2A - 3B = \begin{bmatrix} 2-12 & -4-3 \\ 6+3 & 0-15 \end{bmatrix} = \begin{bmatrix} -10 & -7 \\ 9 & -15 \end{bmatrix}.$$

Example 1.10 — Finding an Unknown Matrix

Find the matrix X such that $2X + A = B$, where $A = \begin{bmatrix} 3 & -1 \\ 0 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 7 & 5 \\ -2 & 6 \end{bmatrix}$.

Isolate X:

$$2X = B - A = \begin{bmatrix} 7-3 & 5+1 \\ -2-0 & 6-4 \end{bmatrix} = \begin{bmatrix} 4 & 6 \\ -2 & 2 \end{bmatrix}.$$

$$X = \frac{1}{2}\begin{bmatrix} 4 & 6 \\ -2 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ -1 & 1 \end{bmatrix}.$$

Properties of Matrix Addition and Scalar Multiplication

Let A, B, C be $m \times n$ matrices and k, ℓ be scalars. Then:

1. $A + B = B + A$ *(Commutativity of Addition)*
2. $(A + B) + C = A + (B + C)$ *(Associativity of Addition)*
3. $A + \mathbf{0} = A$ *(Additive Identity)*
4. $A + (-A) = \mathbf{0}$ *(Additive Inverse)*
5. $k(A + B) = kA + kB$ *(Distributivity I)*
6. $(k + \ell)A = kA + \ell A$ *(Distributivity II)*
7. $k(\ell A) = (k\ell)A$ *(Associativity of Scalar Mult.)*
8. $1 \cdot A = A$ *(Scalar Identity)*

Sigma (Summation) Notation

For a sequence of numbers $a_1, a_2, \ldots, a_n$, the **summation notation** is defined as:

$$\sum_{k=1}^{n} a_k = a_1 + a_2 + \cdots + a_n,$$

where k is the **index of summation**, 1 is the **lower limit** and n is the **upper limit**. This notation is used in the definition of matrix multiplication to express the (i, j)-entry of a product compactly.

Example 1.11 — Evaluating a Summation

Compute $\sum_{k=1}^{4}(2k - 1)$.

$$\sum_{k=1}^{4}(2k - 1) = (2(1) - 1) + (2(2) - 1) + (2(3) - 1) + (2(4) - 1) = 1 + 3 + 5 + 7 = 16.$$

Example 1.12 — Sum of Squares

Compute $\sum_{k=1}^{3} k^2$.

$$\sum_{k=1}^{3} k^2 = 1^2 + 2^2 + 3^2 = 1 + 4 + 9 = 14.$$

Example 1.13 — Writing an Expression in Sigma Notation

Write the expression $a_{11}b_{11} + a_{12}b_{21} + a_{13}b_{31}$ in sigma notation.

$$\sum_{k=1}^{3} a_{1k}\, b_{k1}.$$

This is exactly the $(1,1)$-entry of the matrix product AB, as defined in the next section.

Matrix Multiplication

Let A be an $m \times p$ matrix and B be a $p \times n$ matrix. The **product** AB is the $m \times n$ matrix $C = [c_{ij}]$ whose (i,j)-entry is:

$$c_{ij} = \sum_{k=1}^{p} a_{ik}\, b_{kj} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{ip}b_{pj}.$$

Compatibility Rule: The product AB is defined if and only if the number of **columns of** A equals the number of **rows of** B. If A is $m \times p$ and B is $p \times n$, then AB is $m \times n$:

$$\underbrace{A}_{m\times p}\ \underbrace{B}_{p\times n} = \underbrace{AB}_{m\times n}.$$

Example 1.14 — Product of Two 2×2 Matrices

Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix}$. Compute AB.

Both are 2×2, so AB is 2×2.

$$\begin{aligned} c_{11} &= (1)(5) + (2)(7) = 5 + 14 = 19, \\ c_{12} &= (1)(6) + (2)(8) = 6 + 16 = 22, \\ c_{21} &= (3)(5) + (4)(7) = 15 + 28 = 43, \\ c_{22} &= (3)(6) + (4)(8) = 18 + 32 = 50. \end{aligned}$$

$$AB = \begin{bmatrix} 19 & 22 \\ 43 & 50 \end{bmatrix}.$$

Example 1.15 — Product of Matrices of Different Sizes

Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$ and $B = \begin{bmatrix} 7 & 8 \\ 9 & 10 \\ 11 & 12 \end{bmatrix}$. Compute AB.

A is 2×3 and B is 3×2, so AB is 2×2.

$$\begin{aligned} c_{11} &= (1)(7) + (2)(9) + (3)(11) = 7 + 18 + 33 = 58, \\ c_{12} &= (1)(8) + (2)(10) + (3)(12) = 8 + 20 + 36 = 64, \\ c_{21} &= (4)(7) + (5)(9) + (6)(11) = 28 + 45 + 66 = 139, \\ c_{22} &= (4)(8) + (5)(10) + (6)(12) = 32 + 50 + 72 = 154. \end{aligned}$$

$$AB = \begin{bmatrix} 58 & 64 \\ 139 & 154 \end{bmatrix}.$$

Properties of Matrix Multiplication

Let A, B, C be matrices of sizes for which the operations are defined and k be a scalar. Then:

1. $(AB)C = A(BC)$ *(Associativity)*
2. $A(B + C) = AB + AC$ *(Left Distributivity)*
3. $(A + B)C = AC + BC$ *(Right Distributivity)*
4. $k(AB) = (kA)B = A(kB)$ *(Scalar Associativity)*
5. $AI = IA = A$ *(Identity)*

The following properties do NOT hold in general:

- $AB \neq BA$ *(commutative law fails)*
- $AB = \mathbf{0} \not\Rightarrow A = \mathbf{0}$ or $B = \mathbf{0}$ *(zero-product law fails)*
- $AB = AC \not\Rightarrow B = C$ *(cancellation law fails)*

Example 1.16 -Past Paper 2021 Q1(ii)

Which property does not hold in matrix multiplication? Write one example to justify.

The **commutative property** fails. In general $AB \neq BA$.
Let $A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}$.

$$AB = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 0+1 & 0+1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}.$$

$$BA = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}.$$

Since $AB \neq BA$, the commutative property does **not** hold.

Example 1.17 — Zero Product Without Zero Factors

Give an example of two nonzero matrices A and B such that $AB = \mathbf{0}$.

Let $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$.

$$AB = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \mathbf{0}.$$

Neither A nor B is the zero matrix, yet $AB = \mathbf{0}$. This shows the **zero-product law fails** for matrices.

Example 1.18 — Cancellation Law Fails

Find matrices A, B, C with $A \neq \mathbf{0}$ such that $AB = AC$ but $B \neq C$.

Let $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix}$, $C = \begin{bmatrix} 0 & 3 \\ 1 & 0 \end{bmatrix}$.

$$AB = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 4 & 6 \end{bmatrix}.$$

$$AC = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} 0 & 3 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0+2 & 3+0 \\ 0+4 & 6+0 \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 4 & 6 \end{bmatrix}.$$

So $AB = AC$ yet $B \neq C$, confirming the **cancellation law fails** for matrices.

Summary — Three Laws That Fail for Matrices:

1. $AB \neq BA$ in general *(commutative law fails)*
2. $AB = \mathbf{0}$ does **not** imply $A = \mathbf{0}$ or $B = \mathbf{0}$ *(zero-product law fails)*
3. $AB = AC$ does **not** imply $B = C$ *(cancellation law fails)*

Transpose of a Matrix

The **transpose** of an $m \times n$ matrix $A = [a_{ij}]$ is the $n \times m$ matrix $A^T = [a_{ji}]$ obtained by writing the rows of A as the columns of A^T. That is, the (i, j)-entry of A^T equals the (j, i)-entry of A:

$$(A^T)_{ij} = a_{ji}.$$

Example 1.19 — Computing the Transpose

Find the transpose of $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$.

Solution: A is 2×3, so A^T is 3×2. Rows of A become columns of A^T:

$$A^T = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}.$$

Example 1.20 — Transpose of a Square Matrix

Find A^T for $A = \begin{bmatrix} 5 & -3 & 0 \\ 2 & 1 & 7 \\ -1 & 4 & 6 \end{bmatrix}$.

Solution: Write each row of A as a column:

$$A^T = \begin{bmatrix} 5 & 2 & -1 \\ -3 & 1 & 4 \\ 0 & 7 & 6 \end{bmatrix}.$$

Example 1.21 — Past Paper 2025 Q1(ii) — Transpose and Inverse

If A is a symmetric matrix, show that $(A^{-1})^T = (A^T)^{-1}$.

Solution: Since A is symmetric, $A^T = A$. We need to show that A^{-1} is also symmetric, i.e. $(A^{-1})^T = A^{-1}$.

Start from $AA^{-1} = I$. Taking the transpose of both sides:

$$(AA^{-1})^T = I^T \;\Rightarrow\; (A^{-1})^T A^T = I.$$

Since A is symmetric $A^T = A$, so:

$$(A^{-1})^T A = I.$$

This means $(A^{-1})^T$ is the inverse of A, i.e. $(A^{-1})^T = A^{-1} = (A^T)^{-1}$. □

Properties of the Transpose

Let A and B be matrices and k a scalar. Then:

1. $(A^T)^T = A$
2. $(A+B)^T = A^T + B^T$
3. $(kA)^T = kA^T$
4. $(AB)^T = B^T A^T$ *(note the reversed order)*

Main Diagonal and Trace

In a square matrix A of order n, the entries $a_{11}, a_{22}, \ldots, a_{nn}$ form the **main diagonal**. The **trace** of A, written $\operatorname{tr}(A)$, is the sum of all main diagonal entries:

$$\operatorname{tr}(A) = a_{11} + a_{22} + \cdots + a_{nn} = \sum_{i=1}^{n} a_{ii}.$$

Symmetric and Skew-Symmetric Matrices

A square matrix A is called:

- **Symmetric** if $A^T = A$, i.e. $a_{ij} = a_{ji}$ for all i, j.
- **Skew-Symmetric** (or anti-symmetric) if $A^T = -A$, i.e. $a_{ij} = -a_{ji}$ for all i, j. Note that every diagonal entry of a skew-symmetric matrix must be zero.

Example 1.22 — Symmetric and Skew-Symmetric Matrices

Determine whether each matrix is symmetric, skew-symmetric or neither.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 2 & -3 \\ -2 & 0 & 1 \\ 3 & -1 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}.$$

Solution:

For A: $A^T = A$, so A is **symmetric**.

For B: $B^T = \begin{bmatrix} 0 & -2 & 3 \\ 2 & 0 & -1 \\ -3 & 1 & 0 \end{bmatrix} = -B$, so B is **skew-symmetric**.

For C: $C^T = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} \neq C$ and $\neq -C$, so C is **neither**.

Example 1.23 — Finding the Trace

Find the trace of $A = \begin{bmatrix} 3 & 1 & -2 \\ 0 & 5 & 4 \\ 7 & -1 & 2 \end{bmatrix}$.

Solution: The main diagonal entries are 3, 5 and 2:

$$\operatorname{tr}(A) = 3 + 5 + 2 = 10.$$

Powers of Matrices and Polynomials in Matrices

Powers of a Square Matrix

Let A be a square matrix of order n and let p be a positive integer. The **p-th power** of A is defined as:

$$A^p = \underbrace{A \cdot A \cdots A}_{p \text{ times}}.$$

We also define $A^0 = I_n$. If A is invertible, then for any positive integer p:

$$A^{-p} = (A^{-1})^p = \underbrace{A^{-1} \cdot A^{-1} \cdots A^{-1}}_{p \text{ times}}.$$

Example 1.24 — Computing Powers of a Matrix

Let $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$. Compute A^2 and A^3.

Solution:

$$A^2 = AA = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1+0 & 1+1 \\ 0 & 0+1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}.$$

$$A^3 = A^2 \cdot A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}.$$

Polynomial in a Matrix

If $f(x) = a_0 + a_1x + a_2x^2 + \cdots + a_mx^m$ is a polynomial and A is a square matrix, the **matrix polynomial** $f(A)$ is defined by substituting A for x:

$$f(A) = a_0I + a_1A + a_2A^2 + \cdots + a_mA^m.$$

Note that a_0 becomes a_0I since we cannot add a scalar to a matrix.

Example 1.25 — Past Paper 2024 Q1(ix)and 2025 Q1(vii)

Show that the matrix $A = \begin{bmatrix} 1 & 2 \\ 3 & -4 \end{bmatrix}$ is a zero of $g(x) = x^2 + 3x - 10$.

Solution: We need to show $g(A) = A^2 + 3A - 10I = \mathbf{0}$.

First compute A^2:

$$A^2 = \begin{bmatrix} 1 & 2 \\ 3 & -4 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & -4 \end{bmatrix} = \begin{bmatrix} 1+6 & 2-8 \\ 3-12 & 6+16 \end{bmatrix} = \begin{bmatrix} 7 & -6 \\ -9 & 22 \end{bmatrix}.$$

Now compute $3A$:

$$3A = \begin{bmatrix} 3 & 6 \\ 9 & -12 \end{bmatrix}.$$

Finally:

$$g(A) = A^2 + 3A - 10I = \begin{bmatrix} 7 & -6 \\ -9 & 22 \end{bmatrix} + \begin{bmatrix} 3 & 6 \\ 9 & -12 \end{bmatrix} - \begin{bmatrix} 10 & 0 \\ 0 & 10 \end{bmatrix} = \begin{bmatrix} 7+3-10 & -6+6-0 \\ -9+9-0 & 22-12-10 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \mathbf{0}.$$

Hence A is a zero of $g(x)$. □

Example 1.26 -Past Paper 2024 Q1(x) and 2025 Q1(viii)

If $A = \begin{bmatrix} 1 & -1 \\ a & b \end{bmatrix}$ and $A^2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, find a and b.

Solution: Compute A^2:

$$A^2 = \begin{bmatrix} 1 & -1 \\ a & b \end{bmatrix} \begin{bmatrix} 1 & -1 \\ a & b \end{bmatrix} = \begin{bmatrix} 1-a & -1-b \\ a+ab & -a+b^2 \end{bmatrix}.$$

Set equal to I_2 and equate entries:

$$\begin{aligned} 1-a=1 &\Rightarrow a=0, \\ -1-b=0 &\Rightarrow b=-1, \\ a+ab=0 &\Rightarrow 0+0=0 \quad \checkmark \\ -a+b^2=1 &\Rightarrow 0+1=1 \quad \checkmark \end{aligned}$$

Therefore $\boxed{a=0, \quad b=-1}$.

Invertible (Nonsingular) Matrices

Invertible Matrix

A square matrix A of order n is called **invertible** (or **nonsingular**) if there exists a square matrix B of the same order such that:

$$AB = BA = I_n.$$

The matrix B is called the **inverse** of A and is denoted A^{-1}. A matrix that is not invertible is called **singular**.

Formula for the inverse of a 2×2 matrix: If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $\det(A) = ad - bc \neq 0$, then:

$$A^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

If $\det(A) = 0$, then A has no inverse.

Example 1.27 — Finding the Inverse of a 2×2 Matrix

Find the inverse of $A = \begin{bmatrix} 3 & 1 \\ 5 & 2 \end{bmatrix}$.

Solution: $\det(A) = (3)(2) - (1)(5) = 6 - 5 = 1 \neq 0$, so A is invertible.

$$A^{-1} = \frac{1}{1} \begin{bmatrix} 2 & -1 \\ -5 & 3 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ -5 & 3 \end{bmatrix}.$$

Verification: $AA^{-1} = \begin{bmatrix} 3 & 1 \\ 5 & 2 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ -5 & 3 \end{bmatrix} = \begin{bmatrix} 6-5 & -3+3 \\ 10-10 & -5+6 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I.$ ✓

Example 1.28 — Finding the Inverse by Row Reduction

Find the inverse of $A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 1 & 2 & 4 \end{bmatrix}$.

Solution: Form the augmented matrix $[A \mid I]$ and row reduce until the left side becomes I. The right side will then be A^{-1}.

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 1 & 0 \\ 1 & 2 & 4 & 0 & 0 & 1 \end{array}\right]$$

$R_3 \leftarrow R_3 - R_1$:

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 1 & 0 \\ 0 & 1 & 3 & -1 & 0 & 1 \end{array}\right]$$

$R_3 \leftarrow R_3 - R_2$:

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 & -1 & 1 \end{array}\right]$$

$R_2 \leftarrow R_2 - 2R_3$:

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 2 & 3 & -2 \\ 0 & 0 & 1 & -1 & -1 & 1 \end{array}\right]$$

$R_1 \leftarrow R_1 - R_3$:

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 2 & 1 & -1 \\ 0 & 1 & 0 & 2 & 3 & -2 \\ 0 & 0 & 1 & -1 & -1 & 1 \end{array}\right]$$

$R_1 \leftarrow R_1 - R_2$:

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & -2 & 1 \\ 0 & 1 & 0 & 2 & 3 & -2 \\ 0 & 0 & 1 & -1 & -1 & 1 \end{array}\right]$$

Therefore:

$$A^{-1} = \begin{bmatrix} 0 & -2 & 1 \\ 2 & 3 & -2 \\ -1 & -1 & 1 \end{bmatrix}.$$

Example 1.29 — Past Paper 2024 Q1(vi) and 2025 Q1(v)

If A is an invertible matrix and n is a nonnegative integer, show that $(A^n)^{-1} = (A^{-1})^n$.

Solution: We show that $(A^{-1})^n$ is the inverse of A^n by verifying $A^n \cdot (A^{-1})^n = I$.

$$\begin{aligned} A^n \cdot (A^{-1})^n &= \underbrace{A \cdot A \cdots A}_{n} \cdot \underbrace{A^{-1} \cdot A^{-1} \cdots A^{-1}}_{n} \\ &= A^{n-1} \underbrace{(A \cdot A^{-1})}_{I} (A^{-1})^{n-1} \\ &= A^{n-1}(A^{-1})^{n-1} \\ &= \cdots = AA^{-1} = I. \end{aligned}$$

Since $A^n \cdot (A^{-1})^n = I$, we conclude $(A^n)^{-1} = (A^{-1})^n$. □

Special Types of Square Matrices

Diagonal Matrix

A square matrix A is called a **diagonal matrix** if all entries *off* the main diagonal are zero, i.e. $a_{ij} = 0$ whenever $i \neq j$.

$$D = \begin{bmatrix} d_1 & 0 & \cdots & 0 \\ 0 & d_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & d_n \end{bmatrix} = \operatorname{diag}(d_1, d_2, \ldots, d_n).$$

The inverse of a diagonal matrix (when all $d_i \neq 0$) is:

$$D^{-1} = \operatorname{diag}\left(\frac{1}{d_1}, \frac{1}{d_2}, \ldots, \frac{1}{d_n}\right).$$

Example 1.30 — Past Paper 2021 Q1(xv) — Inverse of a Diagonal Matrix

Write the inverse of the diagonal matrix $D = \begin{bmatrix} -2 & 0 & 0 \\ 0 & 0.25 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ without any solution process.

Solution: Take the reciprocal of each diagonal entry:

$$D^{-1} = \begin{bmatrix} -\frac{1}{2} & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Triangular Matrices

A square matrix A is called:

- **Upper Triangular** if all entries *below* the main diagonal are zero ($a_{ij} = 0$ for $i > j$).
- **Lower Triangular** if all entries *above* the main diagonal are zero ($a_{ij} = 0$ for $i < j$).

Example 1.31 — Identifying Triangular Matrices

Identify each matrix as upper triangular, lower triangular or neither.

$$U = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix}, \quad L = \begin{bmatrix} 2 & 0 & 0 \\ 1 & 3 & 0 \\ 4 & 5 & 7 \end{bmatrix}, \quad A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}.$$

Solution: U has zeros below the diagonal: **upper triangular**.
L has zeros above the diagonal: **lower triangular**.
A has no zeros below or above the diagonal: **neither**.

Orthogonal Matrix

A square matrix P is called **orthogonal** if

$$P^T P = PP^T = I,$$

which is equivalent to saying $P^T = P^{-1}$. The rows (and columns) of an orthogonal matrix form an orthonormal set.

Example 1.32 – Past Paper 2024 Q1(vii) and 2025 Q1(vi)

Show that a matrix P is orthogonal if and only if P^T is orthogonal.

Solution: ($\Rightarrow$) Suppose P is orthogonal, so $P^T P = I$.
We need to show P^T is orthogonal, i.e. $(P^T)^T P^T = I$.
Since $(P^T)^T = P$, we have:

$$(P^T)^T P^T = PP^T = I.$$

The last equality holds because P orthogonal means $PP^T = I$. So P^T is orthogonal.

($\Leftarrow$) Suppose P^T is orthogonal. Then by the same argument applied to P^T, we get $(P^T)^T = P$ is orthogonal. □

Complex Matrices

Conjugate and Hermitian Transpose

Let $A = [a_{ij}]$ be a matrix with complex number entries.

- The **conjugate** of A, written $\bar{A}$, is the matrix obtained by replacing each entry a_{ij} with its complex conjugate $\overline{a_{ij}}$.
- The **Hermitian transpose** (or **conjugate transpose**) of A, written A^* or A^H, is defined as:

$$A^* = \bar{A}^T = \overline{A^T}.$$

  That is, take the transpose *and* conjugate every entry.
- A is called **Hermitian** if $A^* = A$, i.e. $a_{ij} = \overline{a_{ji}}$.
- A is called **Unitary** if $A^*A = AA^* = I$, i.e. $A^* = A^{-1}$.

Example 1.33 - Past Paper 2021 Q1(iii) - Hermitian Transpose

Find the Hermitian transpose of $A = \begin{bmatrix} 2+i & 3 \\ 1-2i & 4i \end{bmatrix}$.

Solution: First take the transpose of A:

$$A^T = \begin{bmatrix} 2+i & 1-2i \\ 3 & 4i \end{bmatrix}.$$

Then conjugate every entry (replace i with $-i$):

$$A^* = \begin{bmatrix} 2-i & 1+2i \\ 3 & -4i \end{bmatrix}.$$

Example 1.34 — Past Paper 2021 Q1(iii) and 2023 Q1(iii)

Define a Hermitian matrix. Show that $H = \begin{bmatrix} 3 & 2-i \\ 2+i & 5 \end{bmatrix}$ is Hermitian.

Solution: A matrix H is **Hermitian** if $H^* = H$.
Compute H^*. First transpose: $H^T = \begin{bmatrix} 3 & 2+i \\ 2-i & 5 \end{bmatrix}$. Then conjugate every entry: $H^* = \begin{bmatrix} 3 & 2-i \\ 2+i & 5 \end{bmatrix} = H$.
Since $H^* = H$, the matrix is **Hermitian**. □

Example 1.35 — Past Paper 2021 Q1(xvi) — Unitary Matrix

Define a unitary matrix. Is $U = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & i \\ i & 1 \end{bmatrix}$ unitary?

Solution: A square matrix U is **unitary** if $U^*U = I$.
Compute U^*:

$$U^* = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -i \\ -i & 1 \end{bmatrix}.$$

Now compute U^*U:

$$U^*U = \frac{1}{2} \begin{bmatrix} 1 & -i \\ -i & 1 \end{bmatrix} \begin{bmatrix} 1 & i \\ i & 1 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1+1 & i-i \\ -i+i & 1+1 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I.$$

Since $U^*U = I$, the matrix U is **unitary**. □

Block (Partitioned) Matrix

A matrix can be divided into smaller submatrices called **blocks** by drawing horizontal and vertical dividing lines through it. The resulting matrix written in terms of its blocks is called a **block matrix** or **partitioned matrix**. Block matrices can be added and multiplied using the same rules as ordinary matrices, treating each block as if it were a single entry, provided the block sizes are compatible.

Example 1.36 — Writing a Matrix in Block Form

Partition the matrix A into four blocks as indicated.

$$A = \left[\begin{array}{cc|c} 1 & 2 & 3 \\ 4 & 5 & 6 \\ \hline 7 & 8 & 9 \end{array}\right] = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}.$$

Identify each block.

Solution: Reading off from the partition lines:

$$A_{11} = \begin{bmatrix} 1 & 2 \\ 4 & 5 \end{bmatrix}, \quad A_{12} = \begin{bmatrix} 3 \\ 6 \end{bmatrix}, \quad A_{21} = \begin{bmatrix} 7 & 8 \end{bmatrix}, \quad A_{22} = \begin{bmatrix} 9 \end{bmatrix}.$$

Determinants

Determinant

The **determinant** is a scalar value associated with every square matrix A of order n, denoted $\det(A)$ or $|A|$.

For a 1×1 matrix $A = [a]$: $\det(A) = a$.

For a 2×2 matrix:

$$\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc.$$

For an $n \times n$ matrix ($n \geq 3$), expand along any row or column using cofactors. Expanding along row i:

$$\det(A) = \sum_{j=1}^{n} a_{ij}\, C_{ij},$$

where $C_{ij} = (-1)^{i+j} M_{ij}$ is the **cofactor** and M_{ij} is the **minor** — the determinant of the submatrix obtained by deleting row i and column j from A.

Minor and Cofactor

Let A be an $n \times n$ matrix.

- The **minor** M_{ij} is the determinant of the $(n-1) \times (n-1)$ submatrix formed by deleting row i and column j from A.
- The **cofactor** is $C_{ij} = (-1)^{i+j} M_{ij}$.
- The sign $(-1)^{i+j}$ follows the checkerboard pattern:

$$\begin{bmatrix} + & - & + & - \\ - & + & - & + \\ + & - & + & - \\ - & + & - & + \end{bmatrix}.$$

Example 1.37 — Determinant of a 2×2 Matrix

Find the determinant of $A = \begin{bmatrix} 5 & -3 \\ 2 & 4 \end{bmatrix}$.

Solution:

$$\det(A) = (5)(4) - (-3)(2) = 20 + 6 = 26.$$

Example 1.38 — Determinant of a 3×3 Matrix

Find the determinant of $A = \begin{bmatrix} 2 & -1 & 3 \\ 0 & 4 & -2 \\ 1 & 3 & 5 \end{bmatrix}$.

Solution: Expand along the first row:

$$\det(A) = 2\begin{vmatrix} 4 & -2 \\ 3 & 5 \end{vmatrix} - (-1)\begin{vmatrix} 0 & -2 \\ 1 & 5 \end{vmatrix} + 3\begin{vmatrix} 0 & 4 \\ 1 & 3 \end{vmatrix}.$$

$$= 2(20+6) + 1(0+2) + 3(0-4) = 2(26) + 1(2) + 3(-4) = 52 + 2 - 12 = 42.$$

Example 1.39 — Expanding Along a Column

Find $\det(A)$ for $A = \begin{bmatrix} 1 & 0 & 2 \\ 3 & 1 & -1 \\ 0 & 2 & 4 \end{bmatrix}$ by expanding along column 2.

Solution: Column 2 entries: $a_{12} = 0$, $a_{22} = 1$, $a_{32} = 2$.

$$\det(A) = 0 \cdot C_{12} + 1 \cdot C_{22} + 2 \cdot C_{32}.$$

$$C_{22} = (+1)\begin{vmatrix} 1 & 2 \\ 0 & 4 \end{vmatrix} = 4 - 0 = 4.$$

$$C_{32} = (-1)\begin{vmatrix} 1 & 2 \\ 3 & -1 \end{vmatrix} = (-1)(-1-6) = 7.$$

$$\det(A) = 0 + 1(4) + 2(7) = 4 + 14 = 18.$$

Example 1.40 — Determinant Using Row of Zeros

Without expanding, explain why $\det\begin{bmatrix} 2 & -1 & 3 \\ 4 & 0 & -2 \\ 4 & -2 & 6 \end{bmatrix} = 0$.

Solution: Observe that row 3 $= 2\times$ row 1. When two rows are proportional (one is a multiple of the other), the determinant is zero. Hence $\det(A) = 0$.

Example 1.41 — Past Paper 2024 Q2(a) & Past Paper 2025 Q2(a) — 4×4 Determinant

Compute the determinant of

$$A = \begin{bmatrix} 1 & 0 & 0 & 3 \\ 2 & 7 & 0 & 6 \\ 0 & 6 & 3 & 0 \\ 7 & 3 & 1 & -5 \end{bmatrix}.$$

Solution: Expand along row 1. Since $a_{12} = 0$ and $a_{13} = 0$, only the first and fourth terms contribute:

$$\det(A) = 1 \cdot C_{11} + 3 \cdot C_{14}.$$

Computing M_{11}: Delete row 1 and column 1:

$$M_{11} = \begin{vmatrix} 7 & 0 & 6 \\ 6 & 3 & 0 \\ 3 & 1 & -5 \end{vmatrix}.$$

Expand along row 1:

$$= 7\begin{vmatrix} 3 & 0 \\ 1 & -5 \end{vmatrix} - 0 + 6\begin{vmatrix} 6 & 3 \\ 3 & 1 \end{vmatrix} = 7(-15) + 6(6-9) = -105 - 18 = -123.$$

$C_{11} = (-1)^2(-123) = -123$.

Computing M_{14}: Delete row 1 and column 4:

$$M_{14} = \begin{vmatrix} 2 & 7 & 0 \\ 0 & 6 & 3 \\ 7 & 3 & 1 \end{vmatrix}.$$

Expand along row 1:

$$= 2\begin{vmatrix} 6 & 3 \\ 3 & 1 \end{vmatrix} - 7\begin{vmatrix} 0 & 3 \\ 7 & 1 \end{vmatrix} + 0 = 2(6-9) - 7(0-21) = -6 + 147 = 141.$$

$C_{14} = (-1)^5(141) = -141$.

$$\det(A) = 1(-123) + 3(-141) = -123 - 423 = \boxed{-546}.$$

Example 1.42 — Determinant of a Triangular Matrix

Find the determinant of $U = \begin{bmatrix} 3 & 1 & -2 \\ 0 & 4 & 5 \\ 0 & 0 & -1 \end{bmatrix}$.

Solution: U is upper triangular, so the determinant equals the product of its diagonal entries:

$$\det(U) = 3 \times 4 \times (-1) = -12.$$

Properties of Determinants

Properties of Determinants

Let A and B be $n \times n$ matrices and k a scalar.

1. $\det(I) = 1$.
2. $\det(A^T) = \det(A)$.
3. $\det(AB) = \det(A) \cdot \det(B)$.
4. $\det(kA) = k^n \det(A)$.
5. $\det(A^{-1}) = \dfrac{1}{\det(A)}$ (if A is invertible).
6. $\det(A^n) = [\det(A)]^n$.
7. If any row or column of A is all zeros, $\det(A) = 0$.
8. If two rows (or columns) are identical or proportional, $\det(A) = 0$.
9. Multiplying a row by k multiplies $\det(A)$ by k.
10. Swapping two rows changes the sign of $\det(A)$.

11. Adding a multiple of one row to another leaves $\det(A)$ unchanged.
12. A is invertible $\Leftrightarrow \det(A) \neq 0$.
13. For triangular A: $\det(A) =$ product of diagonal entries.

Example 1.43 — Applying Determinant Properties

Let A be a 3×3 matrix with $\det(A) = 4$. Find:
(a) $\det(3A)$, (b) $\det(A^{-1})$, (c) $\det(A^T)$, (d) $\det(A^3)$.

Solution:
(a) $\det(3A) = 3^3 \det(A) = 27 \times 4 = 108$.
(b) $\det(A^{-1}) = \dfrac{1}{\det(A)} = \dfrac{1}{4}$.
(c) $\det(A^T) = \det(A) = 4$.
(d) $\det(A^3) = [\det(A)]^3 = 4^3 = 64$.

Example 1.44 — Effect of Row Operations on Determinant

Let $\det \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = 6$. Find the determinant after swapping rows 1 and 2, then multiplying the new row 3 by -3.

Solution: Swapping rows 1 and 2 changes the sign: $\det = -6$.
Multiplying row 3 by -3 multiplies the determinant by -3: $\det = (-3)(-6) = 18$.

Example 1.45 — Past Paper 2021 Q3 — Determinant Equations System

If A and B are 3×3 matrices such that $\det(A^2B^3) = 108$ and $\det(A^3B^2) = 72$, find $\det(2A)$ and $\det(B^{-1})$.

Solution: Let $\det(A) = a$ and $\det(B) = b$.

$$a^2b^3 = 108 \tag{1}$$

$$a^3b^2 = 72 \tag{2}$$

Dividing (1) by (2):

$$\frac{b}{a} = \frac{108}{72} = \frac{3}{2} \;\Rightarrow\; b = \frac{3a}{2}.$$

Substituting into (2):

$$a^3 \cdot \frac{9a^2}{4} = 72 \;\Rightarrow\; 9a^5 = 288 \;\Rightarrow\; a^5 = 32 \;\Rightarrow\; a = 2.$$

So $\det(A) = 2$ and $b = 3$, i.e. $\det(B) = 3$.

$$\det(2A) = 2^3 \cdot \det(A) = 8 \times 2 = \boxed{16}.$$

$$\det(B^{-1}) = \frac{1}{\det(B)} = \frac{1}{3} = \boxed{\tfrac{1}{3}}.$$

Example 1.46 — Past Paper 2023 Q3 — Determinant Equations System

If A and B are 6×6 matrices such that $\det(AB^2) = 72$ and $\det(A^2B^2) = 144$, find $\det(2A)$ and $\det(AB^6)$.

Solution: Let $\det(A) = a$ and $\det(B) = b$.

$$ab^2 = 72 \tag{1}$$

$$a^2b^2 = 144 \tag{2}$$

Dividing (2) by (1):

$$a = \frac{144}{72} = 2.$$

From (1): $2b^2 = 72 \Rightarrow b^2 = 36 \Rightarrow b = 6$.

$$\det(2A) = 2^6 \cdot \det(A) = 64 \times 2 = \boxed{128}.$$

$$\det(AB^6) = \det(A) \cdot [\det(B)]^6 = 2 \times 6^6 = 2 \times 46656 = \boxed{93312}.$$

Example 1.47 — Past Paper 2021 Q1(xi) — Determinant of an Orthogonal Matrix

If A is an orthogonal matrix, show that $\det(A) = \pm 1$.

Solution: Since A is orthogonal, $A^TA = I$. Taking determinants:

$$\det(A^TA) = \det(I) = 1 \;\Rightarrow\; \det(A^T) \cdot \det(A) = 1 \;\Rightarrow\; [\det(A)]^2 = 1 \;\Rightarrow\; \det(A) = \pm 1. \quad \square$$

Example 1.48 — Rank of a Matrix — Past Paper 2021 Q1(xiv)

Find the rank of $A = \begin{bmatrix} 3 & 0 & 2 & 2 \\ -6 & 42 & 24 & 54 \\ 21 & -21 & 0 & -15 \end{bmatrix}$.

Solution: Apply row operations to reduce to echelon form.
$R_2 \leftarrow R_2 + 2R_1$, $R_3 \leftarrow R_3 - 7R_1$:

$$\begin{bmatrix} 3 & 0 & 2 & 2 \\ 0 & 42 & 28 & 58 \\ 0 & -21 & -14 & -29 \end{bmatrix}.$$

$R_3 \leftarrow R_3 + \frac{1}{2}R_2$:

$$\begin{bmatrix} 3 & 0 & 2 & 2 \\ 0 & 42 & 28 & 58 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

There are 2 nonzero rows, so $\text{rank}(A) = \boxed{2}$.

Example 1.49 — Eigenvector Cannot Have Two Eigenvalues — Past Paper 2021 Q1(xiii)

Show that an eigenvector of a square matrix cannot correspond to two distinct eigenvalues.

Solution: Suppose $\mathbf{v} \neq \mathbf{0}$ is an eigenvector of A corresponding to λ_1 and λ_2. Then:

$$A\mathbf{v} = \lambda_1\mathbf{v} \quad \text{and} \quad A\mathbf{v} = \lambda_2\mathbf{v}.$$

Subtracting:

$$(\lambda_1 - \lambda_2)\mathbf{v} = \mathbf{0}.$$

Since $\mathbf{v} \neq \mathbf{0}$, we must have $\lambda_1 - \lambda_2 = 0$, i.e. $\lambda_1 = \lambda_2$. This contradicts the assumption that they are distinct. □

Example 1.50 — Singular vs Invertible via Determinant

Determine whether the matrix $A = \begin{bmatrix} 2 & 3 & 1 \\ 4 & 6 & 2 \\ 1 & 0 & 5 \end{bmatrix}$ is invertible.

Solution: Compute $\det(A)$ by expanding along row 1:

$$= 2\begin{vmatrix} 6 & 2 \\ 0 & 5 \end{vmatrix} - 3\begin{vmatrix} 4 & 2 \\ 1 & 5 \end{vmatrix} + 1\begin{vmatrix} 4 & 6 \\ 1 & 0 \end{vmatrix}$$

$$= 2(30 - 0) - 3(20 - 2) + 1(0 - 6) = 60 - 54 - 6 = 0.$$

Since $\det(A) = 0$, the matrix A is **singular** (not invertible).

Practice Problems

1. Find x, y, z given
$$\begin{bmatrix} 2x+y & z \\ x-y & 3z+1 \end{bmatrix} = \begin{bmatrix} 5 & 4 \\ 1 & 7 \end{bmatrix}.$$

2. Let $A = \begin{bmatrix} 2 & -1 & 3 \\ 0 & 4 & -2 \\ 1 & 1 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 & -1 \\ 2 & 3 & 0 \\ -1 & 1 & 2 \end{bmatrix}$. Compute $3A - 2B$.

3. Find the matrix X such that $2X - 3A = B$, where $A = \begin{bmatrix} 1 & -1 & 0 \\ 2 & 3 & -1 \\ 0 & 1 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 4 & 2 & -2 \\ 0 & 1 & 5 \\ 3 & -1 & 2 \end{bmatrix}$.

4. Let $A = \begin{bmatrix} 1 & 2 & 0 \\ -1 & 3 & 1 \\ 0 & 1 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 0 & 1 \\ 1 & -1 & 3 \\ 4 & 2 & 0 \end{bmatrix}$. Compute AB and BA and confirm $AB \neq BA$.

5. Find A^2 and A^3 for $A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix}$.

6. Show that $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ is a zero of $f(x) = x^2 + 1$.

7. If $A = \begin{bmatrix} 1 & -1 \\ a & b \end{bmatrix}$ and $A^2 = I_2$, find all possible values of a and b.

8. Determine whether each matrix is symmetric, skew-symmetric or neither:
$$A = \begin{bmatrix} 0 & 2 & -3 \\ -2 & 0 & 5 \\ 3 & -5 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 3 & -1 \\ 3 & 0 & 2 \\ -1 & 2 & 4 \end{bmatrix}.$$

9. Find the trace of $A = \begin{bmatrix} 3 & -1 & 2 \\ 5 & 0 & -4 \\ 1 & 7 & -2 \end{bmatrix}$ and verify $\operatorname{tr}(A^T) = \operatorname{tr}(A)$.

10. Find the transpose of $A = \begin{bmatrix} 1 & -2 & 4 \\ 3 & 0 & -1 \\ 2 & 5 & 7 \end{bmatrix}$ and verify $(AB)^T = B^T A^T$ for $B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}$.

11. Find the inverse of $A = \begin{bmatrix} 2 & 1 & -1 \\ 0 & 3 & 2 \\ 1 & 0 & 4 \end{bmatrix}$ using row reduction.

12. Write the inverse of $D = \operatorname{diag}(4, -3, 2, -1)$ immediately without any steps.

13. Determine whether $A = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$ is orthogonal.

14. Find the Hermitian transpose A^* of $A = \begin{bmatrix} 2-3i & 1+i & 0 \\ 4i & -1 & 2-i \end{bmatrix}$.

15. Verify that $H = \begin{bmatrix} 5 & 1-2i & 3i \\ 1+2i & 0 & -i \\ -3i & i & 4 \end{bmatrix}$ is Hermitian.

16. Compute the determinant of $A = \begin{bmatrix} 3 & -2 & 1 \\ 4 & 0 & -3 \\ 1 & 5 & 2 \end{bmatrix}$ by expanding along (a) row 1 and (b) column 2.

17. Find $\det(A)$ for the upper triangular matrix $A = \begin{bmatrix} 4 & 3 & -1 & 2 \\ 0 & -2 & 5 & 1 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$.

18. If A is a 4×4 matrix with $\det(A) = 3$, find $\det(2A)$, $\det(A^{-1})$, $\det(A^4)$ and $\det(A^TA)$.

19. If A and B are 3×3 matrices with $\det(A^2B^3) = 250$ and $\det(A^3B^2) = 200$, find $\det(A)$, $\det(B)$ and $\det(3A)$.

20. Determine whether the matrix $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$ is invertible. Justify using the determinant.

21. Use row operations to find the rank of
$$A = \begin{bmatrix} 2 & 4 & -2 & 6 \\ 1 & 2 & -1 & 3 \\ 3 & 6 & -3 & 9 \end{bmatrix}.$$

22. Show that if A is an $n \times n$ skew-symmetric matrix and n is odd, then $\det(A) = 0$. *Hint: Use* $\det(A^T) = \det(A)$ *and* $A^T = -A$.

23. Let A be a 3×3 orthogonal matrix with $\det(A) = 1$. Find $\det(A^{-1})$ and $\det(2A)$.

24. Verify the property $\det(AB) = \det(A)\det(B)$ for $A = \begin{bmatrix} 1 & 2 & 0 \\ 3 & -1 & 1 \\ 0 & 1 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 0 & 1 \\ -1 & 3 & 0 \\ 1 & 1 & 2 \end{bmatrix}$.

25. If $A = \begin{bmatrix} 2 & 1 \\ 5 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & -1 \\ -5 & 2 \end{bmatrix}$, verify that $AB = I$ and hence write A^{-1} and B^{-1} directly.

Chapter 2

System of Linear Equations

Basic Definitions and Solutions

System of Linear Equations

A **system of linear equations** (or **linear system**) is a collection of m equations in n unknowns $x_1, x_2, \ldots, x_n$ of the form:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m \end{aligned}$$

where $a_{ij} \in \mathbb{R}$ are the **coefficients** and $b_i \in \mathbb{R}$ are the **constant terms**.

A **solution** of the system is an ordered n-tuple $(s_1, s_2, \ldots, s_n)$ such that each equation is satisfied when $x_i = s_i$. The set of all solutions is called the **solution set**.

A system is called:

- **Consistent** if it has at least one solution.
- **Inconsistent** if it has no solution.

Three Possible Outcomes for Any Linear System:

1. **Exactly one solution** (unique solution) — system is consistent.
2. **Infinitely many solutions** — system is consistent.
3. **No solution** — system is inconsistent.

There is no linear system that has exactly two solutions.

Example 2.1 — Identifying Consistent and Inconsistent Systems

Classify each system and state its solution set:

(a) $x+y=5, \quad x-y=1$ (b) $x+y=3, \quad 2x+2y=7$ (c) $x+y=4, \quad 2x+2y=8$

Solution:

(a) Adding the two equations: $2x = 6 \Rightarrow x = 3$, then $y = 5 - 3 = 2$.
Unique solution: $(x, y) = (3, 2)$. System is **consistent** (one solution).

(b) Multiply first equation by 2: $2x + 2y = 6$. But the second equation says $2x + 2y = 7$. Contradiction: $6 \neq 7$. System is **inconsistent** (no solution).

(c) Multiply first equation by 2: $2x + 2y = 8$, which is identical to the second equation. The two equations represent the same line. System is **consistent** (infinitely many solutions): $\{(t, 4 - t) : t \in \mathbb{R}\}$.

Coefficient Matrix and Augmented Matrix

Given the system $Ax = b$, the **coefficient matrix** is the $m \times n$ matrix $A = [a_{ij}]$ formed by the coefficients of the unknowns. The **augmented matrix** is the $m \times (n+1)$ matrix formed by appending the constant column b to A:

$$[A \mid b] = \left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array}\right].$$

All row operations are performed on the augmented matrix to solve the system.

Example 2.2 — Writing Augmented Matrices

Write the augmented matrix for each system:

$$\text{(a)} \begin{cases} 2x - y + 3z = 5 \\ x + 4y - z = 2 \\ 3x + 2y + z = 7 \end{cases} \qquad \text{(b)} \begin{cases} x_1 - x_2 + 2x_3 + x_4 = 0 \\ 3x_1 + 2x_2 \qquad\quad + x_4 = 0 \\ 4x_1 + x_2 + 2x_3 + 2x_4 = 0 \end{cases}$$

Solution:

(a)

$$[A \mid b] = \left[\begin{array}{ccc|c} 2 & -1 & 3 & 5 \\ 1 & 4 & -1 & 2 \\ 3 & 2 & 1 & 7 \end{array}\right].$$

(b) (Homogeneous system — constant column is all zeros)

$$[A \mid 0] = \left[\begin{array}{cccc|c} 1 & -1 & 2 & 1 & 0 \\ 3 & 2 & 0 & 1 & 0 \\ 4 & 1 & 2 & 2 & 0 \end{array}\right].$$

Example 2.3 — Reading a System from its Augmented Matrix

Write the system of equations corresponding to:

$$\left[\begin{array}{ccc|c} 1 & 0 & -3 & 4 \\ 0 & 2 & 1 & -1 \\ 5 & -1 & 0 & 2 \end{array}\right].$$

Solution:

$$\begin{aligned} x_1 \qquad\quad - 3x_3 &= 4 \\ 2x_2 + x_3 &= -1 \\ 5x_1 - x_2 \qquad\quad &= 2 \end{aligned}$$

Equivalent Systems and Elementary Operations

Elementary Row Operations

The three **elementary row operations** (EROs) on an augmented matrix are:

1. **Row Interchange:** Swap two rows. Notation: $R_i \leftrightarrow R_j$
2. **Row Scaling:** Multiply a row by a nonzero scalar $k \neq 0$. Notation: $R_i \leftarrow kR_i$
3. **Row Replacement:** Replace a row by the sum of itself and a scalar multiple of another row. Notation: $R_i \leftarrow R_i + kR_j$

Two systems are called **equivalent** if one can be obtained from the other by a finite sequence of elementary row operations. Equivalent systems have exactly the same solution set.

Example 2.4 — Applying Elementary Row Operations

Apply the given row operations to the matrix:

$$M = \left[\begin{array}{ccc|c} 2 & 4 & -2 & 6 \\ 1 & 3 & 0 & 4 \\ 0 & -1 & 2 & 1 \end{array}\right].$$

Perform: (a) $R_1 \leftrightarrow R_2$, then (b) $R_1 \leftarrow \frac{1}{2}R_1$, then (c) $R_2 \leftarrow R_2 - 2R_1$.

Solution:
Step (a) $R_1 \leftrightarrow R_2$:

$$\left[\begin{array}{ccc|c} 1 & 3 & 0 & 4 \\ 2 & 4 & -2 & 6 \\ 0 & -1 & 2 & 1 \end{array}\right]$$

Step (b) $R_1 \leftarrow \frac{1}{2}R_1$ (no change needed; already done above). Proceed directly to step (c).
Step (c) $R_2 \leftarrow R_2 - 2R_1$:

$$\left[\begin{array}{ccc|c} 1 & 3 & 0 & 4 \\ 0 & -2 & -2 & -2 \\ 0 & -1 & 2 & 1 \end{array}\right].$$

Example 2.5 — Checking Solution Equivalence after Row Operations

Show that the systems

$$\text{(I)} \begin{cases} x + 2y = 5 \\ 3x - y = 1 \end{cases} \qquad \text{and} \qquad \text{(II)} \begin{cases} x + 2y = 5 \\ -7y = -14 \end{cases}$$

are equivalent and have the same solution.

Solution: System (II) is obtained from System (I) by $R_2 \leftarrow R_2 - 3R_1$.
From System (II): $-7y = -14 \Rightarrow y = 2$. Substitute: $x + 4 = 5 \Rightarrow x = 1$.
Verify in System (I):

$$(1) + 2(2) = 5 \checkmark \qquad 3(1) - (2) = 1 \checkmark$$

Both systems have solution $(x, y) = (1, 2)$. □

Example 2.6 — Elementary Operations on a 3×3 System

Use elementary row operations to simplify:

$$\left[\begin{array}{ccc|c} 0 & 2 & -1 & 3 \\ 1 & -1 & 2 & 5 \\ 3 & 1 & 0 & 4 \end{array}\right].$$

Bring to a form where the first column has a leading 1 in row 1.

Solution:

$R_1 \leftrightarrow R_2$:

$$\left[\begin{array}{ccc|c} 1 & -1 & 2 & 5 \\ 0 & 2 & -1 & 3 \\ 3 & 1 & 0 & 4 \end{array}\right]$$

$R_3 \leftarrow R_3 - 3R_1$:

$$\left[\begin{array}{ccc|c} 1 & -1 & 2 & 5 \\ 0 & 2 & -1 & 3 \\ 0 & 4 & -6 & -11 \end{array}\right].$$

The first column now has a leading 1 in row 1 with zeros below it.

Small Square Systems

Cramer's Rule for 2×2 Systems

For the system

$$a_1x + b_1y = c_1, \qquad a_2x + b_2y = c_2,$$

let $D = a_1b_2 - a_2b_1$. If $D \neq 0$, the unique solution is:

$$x = \frac{D_x}{D} = \frac{c_1b_2 - c_2b_1}{D}, \qquad y = \frac{D_y}{D} = \frac{a_1c_2 - a_2c_1}{D},$$

where D_x is obtained by replacing the x-column of D with the constants, and D_y by replacing the y-column. If $D = 0$, the system is either inconsistent or has infinitely many solutions.

Example 2.7 — Cramer's Rule for a 2×2 System

Solve using Cramer's Rule:

$$3x - 2y = 7, \qquad x + 4y = 5.$$

Solution:

$$D = \begin{vmatrix} 3 & -2 \\ 1 & 4 \end{vmatrix} = (3)(4) - (-2)(1) = 12 + 2 = 14.$$

$$D_x = \begin{vmatrix} 7 & -2 \\ 5 & 4 \end{vmatrix} = (7)(4) - (-2)(5) = 28 + 10 = 38. \qquad D_y = \begin{vmatrix} 3 & 7 \\ 1 & 5 \end{vmatrix} = (3)(5) - (7)(1) = 15 - 7 = 8.$$

$$x = \frac{38}{14} = \boxed{\frac{19}{7}}, \qquad y = \frac{8}{14} = \boxed{\frac{4}{7}}.$$

Example 2.8 — Cramer's Rule for a 3×3 System

Solve using Cramer's Rule:

$$x + y + z = 6, \qquad 2x - y + z = 3, \qquad x + 2y - z = 2.$$

Solution:

$$D = \begin{vmatrix} 1 & 1 & 1 \\ 2 & -1 & 1 \\ 1 & 2 & -1 \end{vmatrix}.$$

Expanding along row 1:

$$\begin{aligned} D &= 1\begin{vmatrix} -1 & 1 \\ 2 & -1 \end{vmatrix} - 1\begin{vmatrix} 2 & 1 \\ 1 & -1 \end{vmatrix} + 1\begin{vmatrix} 2 & -1 \\ 1 & 2 \end{vmatrix} \\ &= 1(1-2) - 1(-2-1) + 1(4+1) \\ &= (-1) - (-3) + 5 = -1 + 3 + 5 = 7. \end{aligned}$$

$$D_x = \begin{vmatrix} 6 & 1 & 1 \\ 3 & -1 & 1 \\ 2 & 2 & -1 \end{vmatrix} = 6(1-2) - 1(-3-2) + 1(6+2) = -6 + 5 + 8 = 7.$$

$$D_y = \begin{vmatrix} 1 & 6 & 1 \\ 2 & 3 & 1 \\ 1 & 2 & -1 \end{vmatrix} = 1(-3-2) - 6(-2-1) + 1(4-3) = -5 + 18 + 1 = 14.$$

$$D_z = \begin{vmatrix} 1 & 1 & 6 \\ 2 & -1 & 3 \\ 1 & 2 & 2 \end{vmatrix} = 1(-2-6) - 1(4-3) + 6(4+1) = -8 - 1 + 30 = 21.$$

$$x = \frac{D_x}{D} = \frac{7}{7} = \boxed{1}, \quad y = \frac{D_y}{D} = \frac{14}{7} = \boxed{2}, \quad z = \frac{D_z}{D} = \frac{21}{7} = \boxed{3}.$$

Example 2.9 — Detecting No Unique Solution via Cramer's Rule

Attempt to solve using Cramer's Rule:

$$2x - 4y = 6, \qquad -x + 2y = 3.$$

Solution:

$$D = \begin{vmatrix} 2 & -4 \\ -1 & 2 \end{vmatrix} = (2)(2) - (-4)(-1) = 4 - 4 = 0.$$

Since $D = 0$, Cramer's Rule does not apply. Check consistency: multiply the second equation by -2: $2x - 4y = -6$. But the first equation gives $2x - 4y = 6$. Contradiction $\Rightarrow$ the system is **inconsistent** (no solution).

Systems in Triangular and Echelon Forms

Triangular and Echelon Form

A system is in **upper triangular form** if, in the coefficient matrix, all entries below the main diagonal are zero. For example, a 3×3 triangular system has the shape:

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1 \\ \qquad\qquad a_{22}x_2 + a_{23}x_3 = b_2 \\ \qquad\qquad\qquad\quad a_{33}x_3 = b_3 \end{cases}$$

Such a system is solved by **back substitution**: solve the last equation first, then substitute upward.

A matrix is in **row echelon form (REF)** if:

1. All zero rows are at the bottom.
2. The first nonzero entry in each row (called the **pivot** or **leading entry**) is strictly to the right of the pivot in the row above.
3. All entries below each pivot are zero.

Example 2.10 — Back Substitution in a Triangular System

Solve the triangular system:

$$2x + 3y - z = 9, \qquad 4y + 2z = 6, \qquad 3z = 6.$$

Solution:
From the third equation:

$$3z = 6 \implies z = 2.$$

Substitute $z = 2$ into the second equation:

$$4y + 2(2) = 6 \implies 4y = 2 \implies y = \tfrac{1}{2}.$$

Substitute $y = \frac{1}{2}$, $z = 2$ into the first equation:

$$2x + 3\left(\tfrac{1}{2}\right) - 2 = 9 \implies 2x + \tfrac{3}{2} - 2 = 9 \implies 2x = 9 - \tfrac{3}{2} + 2 = \tfrac{19}{2} \implies x = \tfrac{19}{4}.$$

Solution: $\left(\dfrac{19}{4}, \dfrac{1}{2}, 2\right)$.

Example 2.11 — Identifying Row Echelon Form

Determine which matrices are in row echelon form:

$$(A) = \begin{bmatrix} 1 & 3 & -1 \\ 0 & 2 & 4 \\ 0 & 0 & 5 \end{bmatrix}, \quad (B) = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 3 \\ 0 & 0 & 1 \end{bmatrix}, \quad (C) = \begin{bmatrix} 1 & 2 & 0 & 4 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 2 \end{bmatrix}.$$

Solution:
(A) Pivots are at positions $(1,1)$, $(2,2)$, $(3,3)$, each strictly right of the one above, all below-pivot entries are zero. **Yes, (A) is in REF.**

(B) The pivot of row 2 (position $(2,1)$) is to the *left* of the pivot of row 1 (position $(1,2)$). **No, (B) is not in REF.**

(C) Row 1 pivot at column 1, row 2 pivot at column 3 (column 2 is skipped — allowed), row 3 pivot at column 4. Each pivot is strictly to the right of the one above. **Yes, (C) is in REF.**

Example 2.12 — Solving a System via Back Substitution from REF

Solve the system whose augmented matrix in REF is:

$$\left[\begin{array}{ccc|c} 1 & -2 & 1 & 3 \\ 0 & 1 & 4 & 2 \\ 0 & 0 & 1 & -1 \end{array}\right].$$

Solution:
The system reads:

$$\begin{aligned} x_1 - 2x_2 + x_3 &= 3 \\ x_2 + 4x_3 &= 2 \\ x_3 &= -1. \end{aligned}$$

From row 3: $x_3 = -1$.
From row 2: $x_2 + 4(-1) = 2 \Rightarrow x_2 = 6$.
From row 1: $x_1 - 2(6) + (-1) = 3 \Rightarrow x_1 - 13 = 3 \Rightarrow x_1 = 16$.
Solution: $\boxed{(x_1, x_2, x_3) = (16,\ 6,\ -1)}$.

Gaussian Elimination

Gaussian Elimination

Gaussian elimination is a systematic algorithm that transforms the augmented matrix $[A \mid b]$ into **row echelon form** using elementary row operations, then solves by **back substitution**.

Gauss-Jordan elimination continues further, reducing the matrix to **reduced row echelon form (RREF)**, where each pivot equals 1 and all other entries in its column are zero. The solution can then be read off directly without back substitution.

Steps for Gaussian Elimination:

1. Write the augmented matrix $[A \mid b]$.
2. Identify the leftmost nonzero column (pivot column).
3. Swap rows if needed to place a nonzero entry at the top of the pivot column.
4. Use row replacement operations to create zeros below the pivot.
5. Ignore the top row and repeat steps 2–4 on the remaining submatrix.
6. Once in REF, solve by back substitution.

Consistency check: If a row of the form $[0\ 0\ \cdots\ 0 \mid c]$ with $c \neq 0$ appears, the system is **inconsistent**.

Example 2.13 — Gaussian Elimination (Unique Solution)

Solve by Gaussian elimination:

$$3x_1 + x_2 - x_3 = -4, \qquad x_1 + x_2 - 2x_3 = -4, \qquad -x_1 + 2x_2 - x_3 = 1.$$

Solution:

Augmented matrix:

$$\left[\begin{array}{ccc|c} 3 & 1 & -1 & -4 \\ 1 & 1 & -2 & -4 \\ -1 & 2 & -1 & 1 \end{array}\right]$$

$R_1 \leftrightarrow R_2$ (bring the simplest leading entry to top):

$$\left[\begin{array}{ccc|c} 1 & 1 & -2 & -4 \\ 3 & 1 & -1 & -4 \\ -1 & 2 & -1 & 1 \end{array}\right]$$

$R_2 \leftarrow R_2 - 3R_1$:

$$\left[\begin{array}{ccc|c} 1 & 1 & -2 & -4 \\ 0 & -2 & 5 & 8 \\ -1 & 2 & -1 & 1 \end{array}\right]$$

$R_3 \leftarrow R_3 + R_1$:

$$\left[\begin{array}{ccc|c} 1 & 1 & -2 & -4 \\ 0 & -2 & 5 & 8 \\ 0 & 3 & -3 & -3 \end{array}\right]$$

$R_3 \leftarrow R_3 + \frac{3}{2}R_2$:

$$\left[\begin{array}{ccc|c} 1 & 1 & -2 & -4 \\ 0 & -2 & 5 & 8 \\ 0 & 0 & \frac{3}{2} & 9 \end{array}\right]$$

Back substitution:

$$\frac{3}{2}x_3 = 9 \implies x_3 = 6.$$

$$-2x_2 + 5(6) = 8 \implies -2x_2 = 8 - 30 = -22 \implies x_2 = 11.$$

$$x_1 + 11 - 2(6) = -4 \implies x_1 + 11 - 12 = -4 \implies x_1 = -3.$$

Solution: $\boxed{(x_1,\ x_2,\ x_3) = (-3,\ 11,\ 6)}$.

Example 2.14 — Past Paper 2024 Q4(a) & 2025 Q4(a)

Solve the system by Gaussian elimination:

$$3x_1 + x_2 - x_3 = -4, \qquad x_1 + x_2 - 2x_3 = -4, \qquad -x_1 + 2x_2 - x_3 = 1.$$

Solution: This is the same system solved in Example 2.13. (See above.) Solution: $\boxed{(x_1,\ x_2,\ x_3) = (-3,\ 11,\ 6)}$.

Example 2.15 — Gauss-Jordan Elimination (RREF)

Solve using Gauss-Jordan elimination:

$$x + y - z = 2, \qquad 2x - y + z = 5, \qquad x - 2y + 3z = 0.$$

Solution:
Augmented matrix:

$$\left[\begin{array}{ccc|c} 1 & 1 & -1 & 2 \\ 2 & -1 & 1 & 5 \\ 1 & -2 & 3 & 0 \end{array}\right]$$

$R_2 \leftarrow R_2 - 2R_1$, $R_3 \leftarrow R_3 - R_1$:

$$\left[\begin{array}{ccc|c} 1 & 1 & -1 & 2 \\ 0 & -3 & 3 & 1 \\ 0 & -3 & 4 & -2 \end{array}\right]$$

$R_2 \leftarrow -\frac{1}{3}R_2$:

$$\left[\begin{array}{ccc|c} 1 & 1 & -1 & 2 \\ 0 & 1 & -1 & -\frac{1}{3} \\ 0 & -3 & 4 & -2 \end{array}\right]$$

$R_3 \leftarrow R_3 + 3R_2$:

$$\left[\begin{array}{ccc|c} 1 & 1 & -1 & 2 \\ 0 & 1 & -1 & -\frac{1}{3} \\ 0 & 0 & 1 & -3 \end{array}\right]$$

Now apply upward elimination (Gauss-Jordan):
$R_2 \leftarrow R_2 + R_3$:

$$\left[\begin{array}{ccc|c} 1 & 1 & -1 & 2 \\ 0 & 1 & 0 & -\frac{10}{3} \\ 0 & 0 & 1 & -3 \end{array}\right]$$

$R_1 \leftarrow R_1 + R_3$:

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & -1 \\ 0 & 1 & 0 & -\frac{10}{3} \\ 0 & 0 & 1 & -3 \end{array}\right]$$

$R_1 \leftarrow R_1 - R_2$:

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & \frac{7}{3} \\ 0 & 1 & 0 & -\frac{10}{3} \\ 0 & 0 & 1 & -3 \end{array}\right]$$

Solution: $\boxed{x_1 = \dfrac{7}{3}, \quad x_2 = -\dfrac{10}{3}, \quad x_3 = -3}$.

Example 2.16 — Inconsistent System via Gaussian Elimination

Solve using Gaussian elimination:

$$x + 2y + z = 3, \qquad 2x + 4y + 2z = 7, \qquad x + y + z = 1.$$

Solution:

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 2 & 4 & 2 & 7 \\ 1 & 1 & 1 & 1 \end{array}\right]$$

$R_2 \leftarrow R_2 - 2R_1$:

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \end{array}\right]$$

Row 2 reads $0 = 1$, which is a contradiction. The system is **inconsistent** — no solution exists.

Example 2.17 — System with Infinitely Many Solutions

Solve using Gaussian elimination:

$$x_1 + 2x_2 - x_3 = 3, \qquad 2x_1 + 4x_2 - 2x_3 = 6, \qquad x_1 + x_2 + x_3 = 4.$$

Solution:

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 3 \\ 2 & 4 & -2 & 6 \\ 1 & 1 & 1 & 4 \end{array}\right]$$

$R_2 \leftarrow R_2 - 2R_1$, $R_3 \leftarrow R_3 - R_1$:

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & 1 \end{array}\right]$$

$R_2 \leftrightarrow R_3$:

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 3 \\ 0 & -1 & 2 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$R_2 \leftarrow -R_2$:

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 3 \\ 0 & 1 & -2 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$x_3 = t$ (free variable). From row 2: $x_2 - 2t = -1 \Rightarrow x_2 = 2t - 1$. From row 1: $x_1 + 2(2t - 1) - t = 3 \Rightarrow x_1 + 3t - 2 = 3 \Rightarrow x_1 = 5 - 3t$.

General solution (infinitely many solutions):

$$\boxed{(x_1,\, x_2,\, x_3) = (5 - 3t,\; 2t - 1,\; t), \quad t \in \mathbb{R}.}$$

Echelon Matrices and Row Canonical Form

Reduced Row Echelon Form

A matrix is in **reduced row echelon form** (RREF), also called **row canonical form**, if it satisfies all conditions of REF and additionally:

1. Every pivot (leading entry) equals **1**.
2. Every entry **above and below** each pivot is zero — the entire pivot column is cleared.

Two matrices are **row equivalent** if one can be obtained from the other by a finite sequence of elementary row operations. Row equivalent matrices always have the same solution set.

REF vs. RREF — Quick Comparison:

Property	REF	RREF
Zero rows at bottom	✓	✓
Each pivot right of above	✓	✓
Zeros below each pivot	✓	✓
Each pivot = 1	×	✓
Zeros above each pivot	×	✓

Example 2.18 — Identifying REF and RREF

Classify each matrix as neither, REF only, or RREF:

$$(A) = \begin{bmatrix} 1 & 3 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}, \qquad (B) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad (C) = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 3 \\ 0 & 0 & 1 \end{bmatrix}.$$

Solution:

(A) Pivots at $(1,1)$, $(2,2)$, $(3,3)$ — each to the right of the one above, zeros below each pivot. ✓ REF.

But entry $(1,2) = 3$ is *above* the pivot in column 2, so not all above-pivot entries are zero. **REF only, not RREF.**

(B) Pivots are all 1, each column has zeros everywhere except the pivot position. **This is RREF** (and also REF).

(C) Row 1 pivot is at column 2, but row 2 pivot is at column 1 — column 1 is to the *left* of column 2. **Neither REF nor RREF.**

Example 2.19 — Reducing a Matrix to RREF

Reduce to RREF:

$$A = \begin{bmatrix} 0 & 2 & 4 & 2 \\ 1 & 3 & 5 & 3 \\ 2 & 8 & 10 & 6 \end{bmatrix}.$$

Solution:

$R_1 \leftrightarrow R_2$ (bring nonzero pivot to top of column 1):

$$\begin{bmatrix} 1 & 3 & 5 & 3 \\ 0 & 2 & 4 & 2 \\ 2 & 8 & 10 & 6 \end{bmatrix}$$

$R_3 \leftarrow R_3 - 2R_1$:

$$\begin{bmatrix} 1 & 3 & 5 & 3 \\ 0 & 2 & 4 & 2 \\ 0 & 2 & 0 & 0 \end{bmatrix}$$

$R_3 \leftarrow R_3 - R_2$:

$$\begin{bmatrix} 1 & 3 & 5 & 3 \\ 0 & 2 & 4 & 2 \\ 0 & 0 & -4 & -2 \end{bmatrix}$$

$R_2 \leftarrow \frac{1}{2}R_2,\ R_3 \leftarrow -\frac{1}{4}R_3$:

$$\begin{bmatrix} 1 & 3 & 5 & 3 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & \frac{1}{2} \end{bmatrix}$$

$R_2 \leftarrow R_2 - 2R_3,\ R_1 \leftarrow R_1 - 5R_3$:

$$\begin{bmatrix} 1 & 3 & 0 & \frac{1}{2} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & \frac{1}{2} \end{bmatrix}$$

$R_1 \leftarrow R_1 - 3R_2$:

$$\begin{bmatrix} 1 & 0 & 0 & \frac{1}{2} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & \frac{1}{2} \end{bmatrix} \quad \leftarrow \textbf{RREF}$$

Example 2.20 — Row Equivalence of Two Matrices

Show that the matrices

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 7 \\ 1 & 3 & 5 \end{bmatrix} \qquad \text{and} \qquad B = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

are row equivalent by reducing A to REF and comparing with B.

Solution:

$R_2 \leftarrow R_2 - 2R_1,\ R_3 \leftarrow R_3 - R_1$:

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix}$$

$R_3 \leftarrow R_3 - R_2$:

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

$R_1 \leftarrow R_1 - 2R_2$:

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} = B.$$

Since A reduces to B by elementary row operations, A and B are **row equivalent**. □

Gaussian Elimination Method

Matrix Equation of a Linear System

Any system of m equations in n unknowns can be written compactly as

$$Ax = b,$$

where A is the $m \times n$ **coefficient matrix**, $x = [x_1, x_2, \ldots, x_n]^T$ is the **unknown vector**, and $b = [b_1, b_2, \ldots, b_m]^T$ is the **constant vector**.

The system $Ax = b$ is consistent if and only if b is in the **column space** of A, i.e. b can be written as a linear combination of the columns of A.

Example 2.21 — Writing a System

Write the matrix equation for the system:

$$2x_1 - x_2 + 3x_3 = 5, \qquad x_1 + 4x_2 - x_3 = 2, \qquad 3x_1 + 2x_2 + x_3 = 7.$$

Solution:

Identify the coefficient matrix A, unknown vector x, and constant vector b:

$$A = \begin{bmatrix} 2 & -1 & 3 \\ 1 & 4 & -1 \\ 3 & 2 & 1 \end{bmatrix}, \qquad x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \qquad b = \begin{bmatrix} 5 \\ 2 \\ 7 \end{bmatrix}.$$

The system is equivalent to the single matrix equation $Ax = b$, i.e.:

$$\begin{bmatrix} 2 & -1 & 3 \\ 1 & 4 & -1 \\ 3 & 2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 5 \\ 2 \\ 7 \end{bmatrix}.$$

Consistency Theorem (Rouche-Capelli)

The system $Ax = b$ is consistent if and only if

$$\operatorname{rank}(A) = \operatorname{rank}([A \mid b]).$$

Moreover, if the system is consistent and has n unknowns:

1. If $\operatorname{rank}(A) = n$, the system has a **unique solution**.
2. If $\operatorname{rank}(A) < n$, the system has **infinitely many solutions**, with $n - \operatorname{rank}(A)$ **free variables**.

Example 2.22 — Using Rank to Determine Solution Type

Without solving, determine the solution type of the system:

$$x_1 + 2x_2 + 3x_3 = 4, \qquad 2x_1 + 4x_2 + 6x_3 = 8, \qquad x_1 + x_2 + 2x_3 = 3.$$

Solution:

Form the augmented matrix and row reduce:

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 2 & 4 & 6 & 8 \\ 1 & 1 & 2 & 3 \end{array}\right]$$

$R_2 \leftarrow R_2 - 2R_1$, $R_3 \leftarrow R_3 - R_1$:

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & -1 & -1 \end{array}\right]$$

$R_2 \leftrightarrow R_3$:

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 0 & -1 & -1 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$\text{rank}(A) = 2$, $\text{rank}([A \mid b]) = 2$, $n = 3$.

Since $\text{rank}(A) = \text{rank}([A \mid b]) = 2 < 3$, the system is **consistent** with $3-2 = 1$ free variable. It has **infinitely many solutions**.

Linear Combinations

Solution as a Linear Combination of Columns

The matrix equation $Ax = b$ can be written as:

$$x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots + x_n\mathbf{a}_n = b,$$

where $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n$ are the columns of A.

Therefore, solving $Ax = b$ is equivalent to asking:

"Can b be expressed as a linear combination of the columns of A?"

If yes, the coefficients $x_1, x_2, \ldots, x_n$ form the solution. If no, the system is inconsistent.

Example 2.23 — Past Paper 2024 Q3(a)

Determine whether the vector $v = (3, 3, -4)$ is a linear combination of $u_1 = (1, 2, 3)$, $u_2 = (2, 3, 7)$, $u_3 = (3, 5, 6)$.

Solution:

We need scalars x_1, x_2, x_3 such that $x_1u_1 + x_2u_2 + x_3u_3 = v$.

This gives the system $Ax = v$ where:

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 3 \\ 2 & 3 & 5 & 3 \\ 3 & 7 & 6 & -4 \end{array}\right]$$

$R_2 \leftarrow R_2 - 2R_1$, $R_3 \leftarrow R_3 - 3R_1$:

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 3 \\ 0 & -1 & -1 & -3 \\ 0 & 1 & -3 & -13 \end{array}\right]$$

$R_3 \leftarrow R_3 + R_2$:

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 3 \\ 0 & -1 & -1 & -3 \\ 0 & 0 & -4 & -16 \end{array}\right]$$

Back substitution:

$$-4x_3 = -16 \implies x_3 = 4.$$

$$-x_2 - 4 = -3 \implies x_2 = -1.$$

$$x_1 + 2(-1) + 3(4) = 3 \implies x_1 = 3 - 10 = -7.$$

Verification:

$$-7(1,2,3)+(-1)(2,3,7)+4(3,5,6) = (-7,-14,-21)+(-2,-3,-7)+(12,20,24) = (3,3,-4) \checkmark$$

Yes, $v = (3,3,-4)$ **is** a linear combination of u_1, u_2, u_3.

Example 2.24 — Linear Combination with No Solution

Determine whether $v = (1,2,4)$ is a linear combination of $u_1 = (1,1,2)$ and $u_2 = (2,3,6)$.

Solution:

Set up the augmented matrix:

$$\left[\begin{array}{cc|c} 1 & 2 & 1 \\ 1 & 3 & 2 \\ 2 & 6 & 4 \end{array}\right]$$

$R_2 \leftarrow R_2 - R_1$, $R_3 \leftarrow R_3 - 2R_1$:

$$\left[\begin{array}{cc|c} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 0 & 2 & 2 \end{array}\right]$$

$R_3 \leftarrow R_3 - 2R_2$:

$$\left[\begin{array}{cc|c} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{array}\right]$$

No contradiction row appears. $\text{rank}(A) = \text{rank}([A \mid b]) = 2$.

Back substitution: $x_2 = 1$, $x_1 = 1 - 2(1) = -1$.

So $v = -1 \cdot u_1 + 1 \cdot u_2$. **Yes**, v is a linear combination of u_1 and u_2.

Example 2.25 — Value of k for a Linear Combination

For what value of k is the vector $(1,-2,k)$ in $\mathbb{R}^3$ a linear combination of $(3,0,-2)$ and $(2,-1,-5)$?

Solution:

We need $x_1(3,0,-2) + x_2(2,-1,-5) = (1,-2,k)$.

This gives the system:

$$\left[\begin{array}{cc|c} 3 & 2 & 1 \\ 0 & -1 & -2 \\ -2 & -5 & k \end{array}\right]$$

From row 2: $x_2 = 2$.
From row 1: $3x_1 + 4 = 1 \implies x_1 = -1$.

Substitute into row 3:

$$-2(-1) - 5(2) = k \implies 2 - 10 = k \implies \boxed{k = -8}.$$

Homogeneous Systems

Homogeneous System

A system $Ax = b$ is called **homogeneous** if $b = \mathbf{0}$, i.e. every right-hand side constant is zero:

$$a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n = 0, \quad i = 1, 2, \ldots, m.$$

Every homogeneous system is **always consistent**, since $x = \mathbf{0}$ (called the **trivial solution**) is always a solution.

A homogeneous system has a **nontrivial solution** (i.e. a solution with at least one $x_i \neq 0$) if and only if it has a free variable, which occurs when:

$$\operatorname{rank}(A) < n \quad \text{(number of unknowns)}.$$

In particular, if the system has **more unknowns than equations** ($n > m$), it always has a nontrivial solution.

Structure of Solutions of a Homogeneous System:

1. The trivial solution $x = \mathbf{0}$ always exists.
2. If x_p and x_q are solutions, so is $x_p + x_q$ (closed under addition).
3. If x_p is a solution and $c \in \mathbb{R}$, then cx_p is also a solution (closed under scalar multiplication).
4. The solution set of $Ax = \mathbf{0}$ forms a **subspace** of $\mathbb{R}^n$, called the **null space** of A.

Example 2.26 — Past Paper 2021 Q2

Examine the homogeneous system for a nontrivial solution:

$$x_1 - x_2 + 2x_3 + x_4 = 0, \qquad 3x_1 + 2x_2 \qquad + x_4 = 0, \qquad 4x_1 + x_2 + 2x_3 + 2x_4 = 0.$$

Solution:

We have $m = 3$ equations and $n = 4$ unknowns. Since $n > m$, a nontrivial solution must exist. We find it explicitly.

Augmented matrix (right column all zeros, omitted for brevity):

$$\left[\begin{array}{cccc|c} 1 & -1 & 2 & 1 & 0 \\ 3 & 2 & 0 & 1 & 0 \\ 4 & 1 & 2 & 2 & 0 \end{array}\right]$$

$R_2 \leftarrow R_2 - 3R_1$:

$$\left[\begin{array}{cccc|c} 1 & -1 & 2 & 1 & 0 \\ 0 & 5 & -6 & -2 & 0 \\ 4 & 1 & 2 & 2 & 0 \end{array}\right]$$

$R_3 \leftarrow R_3 - 4R_1$:

$$\left[\begin{array}{cccc|c} 1 & -1 & 2 & 1 & 0 \\ 0 & 5 & -6 & -2 & 0 \\ 0 & 5 & -6 & -2 & 0 \end{array}\right]$$

$R_3 \leftarrow R_3 - R_2$:

$$\left[\begin{array}{cccc|c} 1 & -1 & 2 & 1 & 0 \\ 0 & 5 & -6 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$\operatorname{rank}(A) = 2 < 4$, so there are $4 - 2 = 2$ free variables. Set $x_3 = s$ and $x_4 = t$ (free).

From row 2:

$$5x_2 = 6s + 2t \implies x_2 = \frac{6s + 2t}{5}.$$

From row 1:

$$x_1 = x_2 - 2s - t = \frac{6s + 2t}{5} - 2s - t = \frac{6s + 2t - 10s - 5t}{5} = \frac{-4s - 3t}{5}.$$

General solution (nontrivial solutions exist for any $s, t \neq 0, 0$):

$$\boxed{(x_1, x_2, x_3, x_4) = \frac{s}{5}(-4, 6, 5, 0) + \frac{t}{5}(-3, 2, 0, 5), \quad s, t \in \mathbb{R}.}$$

Example 2.27 — Homogeneous System with Only Trivial Solution

Solve the homogeneous system:

$$x_1 + 2x_2 - x_3 = 0, \qquad 2x_1 + x_2 + x_3 = 0, \qquad x_1 - x_2 + 2x_3 = 0.$$

Solution:

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 0 \\ 2 & 1 & 1 & 0 \\ 1 & -1 & 2 & 0 \end{array}\right]$$

$R_2 \leftarrow R_2 - 2R_1$, $R_3 \leftarrow R_3 - R_1$:

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 0 \\ 0 & -3 & 3 & 0 \\ 0 & -3 & 3 & 0 \end{array}\right]$$

$R_3 \leftarrow R_3 - R_2$:

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 0 \\ 0 & -3 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$\operatorname{rank}(A) = 2 < 3$, so one free variable exists. Set $x_3 = t$.

From row 2: $-3x_2 + 3t = 0 \implies x_2 = t$.
From row 1: $x_1 + 2t - t = 0 \implies x_1 = -t$.

General solution:

$$(x_1, x_2, x_3) = t(-1,\ 1,\ 1), \quad t \in \mathbb{R}.$$

The system has infinitely many nontrivial solutions; the **only trivial solution** occurs at $t = 0$.

Example 2.28 — Echelon Form of a System (Past Paper 2023 Q1iv)

Convert the coefficient matrix of the given system to echelon form:

$$x_1 - x_2 + 2x_3 = 0, \qquad 4x_1 + x_2 + 2x_3 = 1, \qquad x_1 + x_2 + x_3 = -1.$$

Solution:

Augmented matrix:

$$\left[\begin{array}{ccc|c} 1 & -1 & 2 & 0 \\ 4 & 1 & 2 & 1 \\ 1 & 1 & 1 & -1 \end{array}\right]$$

$R_2 \leftarrow R_2 - 4R_1$, $R_3 \leftarrow R_3 - R_1$:

$$\left[\begin{array}{ccc|c} 1 & -1 & 2 & 0 \\ 0 & 5 & -6 & 1 \\ 0 & 2 & -1 & -1 \end{array}\right]$$

$R_3 \leftarrow 5R_3 - 2R_2$:

$$\left[\begin{array}{ccc|c} 1 & -1 & 2 & 0 \\ 0 & 5 & -6 & 1 \\ 0 & 0 & 7 & -7 \end{array}\right] \leftarrow \textbf{Row Echelon Form}$$

Back substitution from row 3: $7x_3 = -7 \implies x_3 = -1$.
From row 2: $5x_2 + 6 = 1 \implies x_2 = -1$.
From row 1: $x_1 + 1 - 2 = 0 \implies x_1 = 1$.

Solution: $\boxed{(x_1, x_2, x_3) = (1, -1, -1)}$.

Null Space of a Matrix

Null Space

The **null space** (or **kernel**) of an $m \times n$ matrix A is the set of all solutions to the homogeneous system $Ax = \mathbf{0}$:

$$\text{Null}(A) = \{x \in \mathbb{R}^n : Ax = \mathbf{0}\}.$$

The null space is always a **subspace** of $\mathbb{R}^n$.

The **nullity** of A is the dimension of the null space:

$$\text{nullity}(A) = n - \text{rank}(A).$$

This is also the number of free variables in the solution of $Ax = \mathbf{0}$.

Rank-Nullity Theorem: For any $m \times n$ matrix A:

$$\text{rank}(A) + \text{nullity}(A) = n.$$

The rank counts the pivot variables; the nullity counts the free variables. Together they always sum to the total number of unknowns n.

Example 2.29 — Finding the Null Space of a Matrix

Find the null space and nullity of

$$A = \begin{bmatrix} 1 & 2 & -1 & 3 \\ 2 & 4 & 1 & 1 \\ 3 & 6 & 0 & 4 \end{bmatrix}.$$

Solution:

Row reduce the augmented matrix $[A \mid \mathbf{0}]$:

$$\left[\begin{array}{cccc|c} 1 & 2 & -1 & 3 & 0 \\ 2 & 4 & 1 & 1 & 0 \\ 3 & 6 & 0 & 4 & 0 \end{array}\right]$$

$R_2 \leftarrow R_2 - 2R_1$, $R_3 \leftarrow R_3 - 3R_1$:

$$\left[\begin{array}{cccc|c} 1 & 2 & -1 & 3 & 0 \\ 0 & 0 & 3 & -5 & 0 \\ 0 & 0 & 3 & -5 & 0 \end{array}\right]$$

$R_3 \leftarrow R_3 - R_2$:

$$\left[\begin{array}{cccc|c} 1 & 2 & -1 & 3 & 0 \\ 0 & 0 & 3 & -5 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Pivot columns: 1 and 3. Free variables: $x_2 = s$, $x_4 = t$.

From row 2: $3x_3 = 5t \implies x_3 = \frac{5t}{3}$.
From row 1:

$$x_1 = -2s + x_3 - 3t = -2s + \frac{5t}{3} - 3t = -2s - \frac{4t}{3}.$$

General solution:

$$x = s \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} -\frac{4}{3} \\ 0 \\ \frac{5}{3} \\ 1 \end{bmatrix}, \quad s, t \in \mathbb{R}.$$

Null(A) is spanned by these two vectors. nullity$(A) = 2$, rank$(A) = 2$, and $2 + 2 = 4 = n$.
✓

Example 2.30 — Null Space of a Square Matrix

Find the null space of

$$A = \begin{bmatrix} 2 & -4 & 2 \\ 1 & -2 & 1 \\ 3 & -6 & 3 \end{bmatrix}.$$

Solution:

All rows are multiples of $(1, -2, 1)$, so after row reduction:

$$\left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

One pivot variable: x_1. Free variables: $x_2 = s$, $x_3 = t$.
From row 1: $x_1 = 2s - t$.

General solution:

$$x = s \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \quad s, t \in \mathbb{R}.$$

$\text{nullity}(A) = 2$, $\text{rank}(A) = 1$, and $1 + 2 = 3 = n$. ✓

Example 2.31 — Rank and Nullity Verification

For the matrix

$$A = \begin{bmatrix} 1 & 0 & -3 & 0 & 2 \\ 0 & 1 & 5 & 0 & -1 \\ 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

identify the pivot and free variables, state $\text{rank}(A)$ and $\text{nullity}(A)$, and verify the Rank-Nullity Theorem.

Solution:

The matrix is already in RREF. Pivots appear in columns 1, 2, and 4.

$$\text{rank}(A) = 3.$$

Free variables are x_3 and x_5 (columns 3 and 5).

$$\text{nullity}(A) = 5 - 3 = 2.$$

Rank-Nullity check:

$$\text{rank}(A) + \text{nullity}(A) = 3 + 2 = 5 = n. ✓$$

Practice Problems

1. Determine whether the following system is consistent. If consistent, state whether the solution is unique or involves free variables:

$$2x_1 - x_2 + 3x_3 = 4, \quad 4x_1 - 2x_2 + 6x_3 = 9, \quad x_1 + x_2 - x_3 = 1.$$

2. Write the augmented matrix for the system and use Gaussian elimination to solve:

$$x_1 + 2x_2 - x_3 + x_4 = 2, \quad 2x_1 - x_2 + 3x_3 - x_4 = 5, \quad x_1 + 3x_2 - 2x_3 + 2x_4 = 1.$$

3. Reduce the matrix to RREF and identify all pivot and free variables:

$$A = \begin{bmatrix} 0 & 2 & -4 & 6 \\ 1 & 3 & 5 & -2 \\ 2 & 8 & 0 & 4 \\ 1 & 5 & -1 & 2 \end{bmatrix}.$$

4. Use Cramer's Rule to solve:

$$2x - y + 3z = 9, \quad x + 2y - z = 2, \quad 3x - y + 2z = 10.$$

5. Solve by back substitution:

$$3x_1 - 6x_2 + 2x_3 - x_4 = 5, \quad 4x_2 - x_3 + 2x_4 = 3, \quad 2x_3 - 3x_4 = 4, \quad 5x_4 = 10.$$

6. Determine all values of k for which the system has (i) no solution, (ii) a unique solution, (iii) infinitely many solutions:

$$x + 2y + kz = 3, \quad 2x + ky + 8z = 10, \quad 3x + 6y + 12z = k.$$

7. Show that the following homogeneous system has only the trivial solution:

$$x_1 + 2x_2 - x_3 = 0, \quad 2x_1 + 5x_2 + 3x_3 = 0, \quad x_1 + 3x_2 + 4x_3 = 0.$$

8. Find the null space and nullity of:

$$A = \begin{bmatrix} 1 & 3 & -2 & 0 & 2 \\ 0 & 0 & 1 & 4 & -3 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Verify the Rank-Nullity Theorem.

9. Examine the homogeneous system for nontrivial solutions and express the general solution in vector form:

$$x_1 - 2x_2 + x_3 - x_4 = 0, \quad 2x_1 - 4x_2 + 3x_3 + x_4 = 0, \quad 3x_1 - 6x_2 + 4x_3 = 0.$$

10. For what values of h and k is the system consistent?

$$x_1 - 3x_2 = 1, \quad 2x_1 + hx_2 = k.$$

11. Determine whether the following system has a unique solution, no solution, or infinitely many solutions. If a solution exists, find it:

$$x + y + z + w = 4, \quad 2x + y - z + w = 1, \quad x - y + 2z - w = 2, \quad 3x + 2y + z + 2w = 7.$$

12. Find the RREF of the matrix and use it to solve the associated homogeneous system:

$$A = \begin{bmatrix} 1 & -1 & 2 & -1 \\ 2 & -2 & 4 & -2 \\ 1 & 0 & 3 & 1 \\ 0 & 1 & -1 & 3 \end{bmatrix}.$$

13. Two systems share the same coefficient matrix A. System I has solution $(1, -1, 2)$ and System II has solution $(0, 2, -1)$. Write down three other solutions to each system and explain your reasoning.

14. Solve the system using Gauss-Jordan elimination:

$$x_1 - 2x_2 + x_3 - x_4 = 3, \quad 2x_1 - 4x_2 + 2x_3 + x_4 = 5, \quad x_1 - 2x_2 + x_3 + 4x_4 = 7.$$

15. For the matrix

$$A = \begin{bmatrix} 3 & 0 & 2 & 2 \\ -6 & 42 & 24 & 54 \\ 21 & -21 & 0 & -15 \end{bmatrix},$$

find $\text{rank}(A)$ and $\text{nullity}(A)$.

16. Determine whether the vector $v = (2, -1, 3, 5)$ is a linear combination of:

$$u_1 = (1, 0, 1, 2), \quad u_2 = (0, 1, -1, 1), \quad u_3 = (1, 1, 0, 3).$$

17. Find all values of a and b such that the vectors $(1, a, 2)$, $(2, 1, b)$, $(3, 2, 1)$ are linearly dependent in $\mathbb{R}^3$.

18. A 3×5 homogeneous system has $\text{rank}(A) = 2$. (a) Find $\text{nullity}(A)$. (b) Write the general form of the solution. (c) State the minimum number of free variables.

19. Find the LU decomposition of:

$$A = \begin{bmatrix} 2 & 4 & -2 \\ 4 & 9 & -3 \\ -2 & -3 & 7 \end{bmatrix}.$$

Use it to solve $Ax = b$ where $b = (2, 8, 10)^T$.

20. Show that the following system is inconsistent:

$$x_1 + 2x_2 - x_3 = 3, \quad 2x_1 + 4x_2 - 2x_3 = 5, \quad x_1 + 3x_2 + x_3 = 4.$$

21. Find an elementary matrix E such that EA equals the matrix obtained from

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

by the operation $R_2 \leftarrow R_2 - 4R_1$. Then compute E^{-1} and interpret its row operation.

22. Express $A = \begin{bmatrix} 2 & 1 & 1 \\ 4 & 3 & 3 \\ 8 & 7 & 9 \end{bmatrix}$ as a product of elementary matrices.

23. Solve the system $Ax = b$ using LU decomposition, where:

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 5 & -1 \\ 4 & 10 & -1 \end{bmatrix}, \qquad b = \begin{bmatrix} 1 \\ 5 \\ 11 \end{bmatrix}.$$

24. Let A be a 4×6 matrix with $\text{rank}(A) = 3$. (a) How many free variables does $Ax = 0$ have? (b) Is the system $Ax = b$ always consistent for every b? Explain. (c) Can $Ax = b$ have a unique solution? Explain.

25. Given the system of equations with parameter λ:

$$(\lambda - 2)x + y = 0, \quad x + (\lambda - 2)y = 0.$$

Find all values of λ for which the system has a nontrivial solution, and find those solutions.

Chapter 3

Vectorr Spaces

Vector Spaces

Vector Space

Let V be a nonempty set equipped with two operations: **vector addition** (denoted $u+v$ for $u, v \in V$) and **scalar multiplication** (denoted kv for $k \in K$, $v \in V$, where K is a field such as $\mathbb{R}$ or $\mathbb{C}$). Then V is called a **vector space** over K if the following axioms hold for all $u, v, w \in V$ and all $a, b \in K$:

Addition Axioms:

1. $(u+v)+w = u+(v+w)$
2. There exists $0 \in V$ such that $u+0=u$
3. For each $u \in V$ there exists $-u \in V$ such that $u+(-u)=0$
4. $u+v=v+u$

Scalar Multiplication Axioms:

5. $a(u+v) = au+av$
6. $(a+b)u = au+bu$
7. $(ab)u = a(bu)$
8. $1u = u$

The elements of V are called **vectors** and the elements of K are called **scalars**.

Example 3.1 — Verifying the Zero Vector Space

Show that $V = \{0\}$ consisting of a single element 0 with $0+0=0$ and $k0=0$ for every scalar k is a vector space.

Solution: All eight axioms reduce to trivial statements involving only 0:

$$\begin{aligned}
(0+0)+0 = 0+(0+0) &= 0 \\
0+0 &= 0 \quad \text{(so 0 is the zero vector)} \\
0+(-0) = 0+0 &= 0 \quad (\text{so } -0=0) \\
a(0+0) = a0+a0 &= 0
\end{aligned}$$

Every axiom is satisfied since there is only one vector and it behaves consistently under all operations. Hence $V = \{0\}$ is a vector space, called the **zero vector space**. $\square$

Example 3.2 — Verifying $\mathbb{R}^n$ as a Vector Space

Show that $\mathbb{R}^3$ with the usual componentwise addition and scalar multiplication satisfies the vector space axioms, using $u = (1, 2, 3)$, $v = (0, -1, 4)$, $w = (2, 0, 1)$ and scalar $a = 2$.

Solution: Commutativity:

$$u + v = (1, 2, 3) + (0, -1, 4) = (1, 1, 7)$$

$$v + u = (0, -1, 4) + (1, 2, 3) = (1, 1, 7)$$

So $u + v = v + u$. ✓

Zero vector: $0 = (0, 0, 0)$ satisfies $u + 0 = (1, 2, 3) + (0, 0, 0) = (1, 2, 3) = u$. ✓

Distributivity:

$$a(u + v) = 2(1, 1, 7) = (2, 2, 14)$$
$$au + av = 2(1, 2, 3) + 2(0, -1, 4) = (2, 4, 6) + (0, -2, 8) = (2, 2, 14)$$

So $a(u + v) = au + av$. ✓

Since all axioms hold for arbitrary vectors under the standard operations, $\mathbb{R}^3$ (and in general $\mathbb{R}^n$) is a vector space over $\mathbb{R}$. $\square$

Example 3.3 — A Set That Is Not a Vector Space

Let $V = \mathbb{R}^2$ with the usual addition, but define scalar multiplication by

$$k(x, y) = (kx, 0).$$

Show that V is not a vector space under this operation.

Solution: Check Axiom 8, which requires $1u = u$ for all $u \in V$.
Let $u = (3, 5)$. Then:

$$1u = 1(3, 5) = (1 \cdot 3, 0) = (3, 0) \neq (3, 5) = u.$$

Since $1u \neq u$, Axiom 8 fails. Therefore V with this modified scalar multiplication is **not** a vector space. $\square$

Checking Whether a Set Is a Vector Space:

1. It is enough to find **one** axiom that fails to prove a set is not a vector space.
2. To prove a set **is** a vector space, all eight axioms must be verified in general (using arbitrary vectors and scalars, not just specific numbers).
3. The most commonly violated axioms in exam questions are closure, the existence of a zero vector, and Axiom 8 ($1u = u$).

Examples of Vector Spaces

Space K^n

The Space K^n

Let K be a field ($\mathbb{R}$ or $\mathbb{C}$). The set of all n-tuples of elements of K, denoted K^n, is a vector space over K where addition and scalar multiplication are defined componentwise:

$$(a_1, \dots, a_n) + (b_1, \dots, b_n) = (a_1 + b_1, \dots, a_n + b_n),$$

$$k(a_1, \dots, a_n) = (ka_1, \dots, ka_n).$$

The zero vector is $(0, 0, \dots, 0)$.

Example 3.4 — Operations in $\mathbb{R}^4$

Let $u = (1, -2, 0, 3)$ and $v = (2, 1, -4, 5)$ in $\mathbb{R}^4$. Find $3u - 2v$.

Solution:

$$\begin{aligned} 3u &= 3(1, -2, 0, 3) = (3, -6, 0, 9) \\ 2v &= 2(2, 1, -4, 5) = (4, 2, -8, 10) \\ 3u - 2v &= (3, -6, 0, 9) - (4, 2, -8, 10) = (-1, -8, 8, -1). \end{aligned}$$

$$3u - 2v = \boxed{(-1, -8, 8, -1)}$$

Example 3.5 — Complex Space $\mathbb{C}^2$

Let $u = (1 - i, 2)$ and $v = (i, 3 + i)$ in $\mathbb{C}^2$. Find $u + v$ and $(2 + i)u$.

Solution:

$$\begin{aligned} u + v &= (1 - i + i,\ 2 + 3 + i) = (1,\ 5 + i) \\ (2 + i)u &= \big((2 + i)(1 - i),\ (2 + i)(2)\big) \\ &= \big(2 - 2i + i - i^2,\ 4 + 2i\big) \\ &= \big(2 - i + 1,\ 4 + 2i\big) \\ &= (3 - i,\ 4 + 2i). \end{aligned}$$

$$u + v = \boxed{(1,\ 5 + i)}, \qquad (2 + i)u = \boxed{(3 - i,\ 4 + 2i)}$$

Polynomial Space

Polynomial Space $P(t)$

Let $P(t)$ denote the set of all polynomials in a variable t with coefficients from a field K:

$$p(t) = a_0 + a_1 t + a_2 t^2 + \cdots + a_s t^s.$$

$P(t)$ is a vector space over K under the usual addition of polynomials and multiplication of a polynomial by a scalar. The zero polynomial is the zero vector. The subset $P_n(t)$ consisting of all polynomials of degree $\leq n$ is also a vector space.

Example 3.6 — Operations in $P_2(t)$

Let $p(t) = 2t^2 - 3t + 1$ and $q(t) = -t^2 + 4t + 5$ in $P_2(t)$. Find $p(t) + q(t)$ and $3p(t)$.

Solution:

$$\begin{aligned} p(t) + q(t) &= (2t^2 - 3t + 1) + (-t^2 + 4t + 5) \\ &= (2 - 1)t^2 + (-3 + 4)t + (1 + 5) \\ &= t^2 + t + 6. \end{aligned}$$

$$3p(t) = 3(2t^2 - 3t + 1) = 6t^2 - 9t + 3.$$

$$p(t) + q(t) = \boxed{t^2 + t + 6}, \qquad 3p(t) = \boxed{6t^2 - 9t + 3}$$

Matrix Space

Matrix Space $M_{m,n}$

The set of all $m \times n$ matrices with entries from a field K, denoted $M_{m,n}(K)$ or simply $M_{m,n}$, forms a vector space over K under the usual matrix addition and scalar multiplication. The zero matrix is the zero vector.

Example 3.7 — Operations in $M_{2,2}$

Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & -1 \\ 2 & 1 \end{bmatrix}$ in $M_{2,2}$. Find $2A - 3B$.

Solution:

$$2A = \begin{bmatrix} 2 & 4 \\ 6 & 8 \end{bmatrix}, \qquad 3B = \begin{bmatrix} 0 & -3 \\ 6 & 3 \end{bmatrix}$$

$$2A - 3B = \begin{bmatrix} 2 & 4 \\ 6 & 8 \end{bmatrix} - \begin{bmatrix} 0 & -3 \\ 6 & 3 \end{bmatrix} = \begin{bmatrix} 2 & 7 \\ 0 & 5 \end{bmatrix}.$$

$$2A - 3B = \boxed{\begin{bmatrix} 2 & 7 \\ 0 & 5 \end{bmatrix}}$$

Function Space

Function Space $F(X)$

Let X be a nonempty set and K a field. The set $F(X)$ of all functions $f : X \to K$ forms a vector space over K, where for $f, g \in F(X)$ and $k \in K$:

$$(f + g)(x) = f(x) + g(x), \qquad (kf)(x) = k\,f(x) \quad \text{for all } x \in X.$$

The zero vector is the zero function, which sends every $x \in X$ to 0.

Example 3.8 — Operations in a Function Space

Let $f(x) = \sin x$ and $g(x) = x^2$ be functions in $F(\mathbb{R})$. Describe $(f + g)(x)$ and $(3f)(x)$.

Solution: By definition of addition and scalar multiplication in a function space:

$$(f + g)(x) = f(x) + g(x) = \sin x + x^2.$$

$$(3f)(x) = 3f(x) = 3\sin x.$$

$$(f+g)(x) = \boxed{\sin x + x^2}, \qquad (3f)(x) = \boxed{3\sin x}$$

Common Vector Spaces to Remember:

1. K^n — n-tuples with componentwise operations
2. $P(t)$ and $P_n(t)$ — polynomials (all degrees, or degree $\leq n$)
3. $M_{m,n}$ — $m \times n$ matrices
4. $F(X)$ — functions from a set X into a field K
5. In every case, the zero vector is the "zero object" of that space (zero tuple, zero polynomial, zero matrix, zero function).

Linear Combinations and Spanning Sets

Linear Combinations

Linear Combination

Let V be a vector space over a field K, and let $u_1, u_2, \dots, u_m \in V$. A vector $v \in V$ is called a **linear combination** of $u_1, u_2, \dots, u_m$ if there exist scalars $a_1, a_2, \dots, a_m \in K$ such that

$$v = a_1u_1 + a_2u_2 + \cdots + a_mu_m.$$

Finding such scalars (if they exist) reduces to solving a system of linear equations.

Example 3.9 — Expressing a Vector as a Linear Combination

Write $v = (4,9)$ as a linear combination of $u_1 = (1,2)$ and $u_2 = (2,3)$.

Solution: We seek scalars a, b such that $v = au_1 + bu_2$:

$$a(1,2) + b(2,3) = (4,9)$$
$$a + 2b = 4$$
$$2a + 3b = 9$$

Multiply the first equation by 2 and subtract from the second:

$$(2a+3b) - (2a+4b) = 9 - 8$$
$$-b = 1 \implies b = -1$$

Substitute back: $a + 2(-1) = 4 \implies a = 6$.

Verification: $6(1,2) - 1(2,3) = (6,12) + (-2,-3) = (4,9)$. ✓

$$v = \boxed{6u_1 - u_2}$$

Example 3.10 — Past Paper 2024 Q1(iii) & 2025 Q1(iii)

For what value of k is the vector $(1,-2,k)$ in $\mathbb{R}^3$ a linear combination of $(3,0,-2)$ and $(2,-1,-5)$?

Solution: We require scalars a, b such that

$$a(3,0,-2)+b(2,-1,-5)=(1,-2,k).$$

This gives the system:

$$\begin{aligned} 3a+2b&=1 \\ -b&=-2 \\ -2a-5b&=k \end{aligned}$$

From the second equation, $b=2$. Substituting into the first:

$$3a+2(2)=1 \implies 3a=-3 \implies a=-1.$$

Substitute $a=-1$, $b=2$ into the third equation:

$$k=-2(-1)-5(2)=2-10=-8.$$

$$k=\boxed{-8}$$

Example 3.11 — Past Paper 2024 Q3(a)

Determine whether $v=(3,3,-4)$ is a linear combination of $x=(1,2,3)$, $y=(2,3,7)$, and $z=(3,5,6)$.

Solution: Seek scalars a, b, c such that $ax+by+cz=v$:

$$\begin{aligned} a+2b+3c&=3 && (1) \\ 2a+3b+5c&=3 && (2) \\ 3a+7b+6c&=-4 && (3) \end{aligned}$$

Compute $(2)-2(1)$:

$$-b-c=-3 \implies b+c=3 \tag{4}$$

Compute $(3)-3(1)$:

$$b-3c=-13 \tag{5}$$

From (4), $b=3-c$. Substitute into (5):

$$\begin{aligned} (3-c)-3c&=-13 \\ 3-4c&=-13 \\ c&=4 \end{aligned}$$

Then $b=3-4=-1$, and from (1): $a=3-2(-1)-3(4)=3+2-12=-7$.

Verification (Eq. 3): $3(-7)+7(-1)+6(4)=-21-7+24=-4$. ✓

Since a consistent solution exists, v **is** a linear combination of x, y, z:

$$v=\boxed{-7x-y+4z}$$

Spanning Sets

Linear Span and Spanning Set

Let $S = \{u_1, u_2, \ldots, u_m\}$ be a subset of a vector space V. The set of all linear combinations of vectors in S is called the **linear span** of S, denoted $\text{span}(S)$:

$$\text{span}(S) = \{a_1u_1 + a_2u_2 + \cdots + a_mu_m : a_i \in K\}.$$

If $\text{span}(S) = V$, we say that S **spans** V, or that S is a **spanning set** of V.

Example 3.12 — Verifying a Spanning Set of $\mathbb{R}^3$

Show that $S = \{(1, 1, 1), (1, 1, 0), (1, 0, 0)\}$ spans $\mathbb{R}^3$.

Solution: Let (a, b, c) be an arbitrary vector in $\mathbb{R}^3$. We must find scalars x, y, z such that

$$x(1, 1, 1) + y(1, 1, 0) + z(1, 0, 0) = (a, b, c).$$

This gives the system:

$$\begin{aligned} x + y + z &= a \\ x + y &= b \\ x &= c \end{aligned}$$

From the third equation, $x = c$. Substituting into the second: $y = b - c$. Substituting both into the first: $z = a - b$.

Since we obtained a solution for **arbitrary** (a, b, c):

$$(a, b, c) = c(1, 1, 1) + (b - c)(1, 1, 0) + (a - b)(1, 0, 0),$$

every vector in $\mathbb{R}^3$ is a linear combination of S. Hence S spans $\mathbb{R}^3$. $\square$

Example 3.13 — Spanning Set of a Polynomial Space

Show that $S = \{1, t, t^2\}$ spans $P_2(t)$.

Solution: Any polynomial in $P_2(t)$ has the general form

$$p(t) = c_0 + c_1t + c_2t^2$$

for some scalars c_0, c_1, c_2. This is already exactly a linear combination of 1, t, and t^2:

$$p(t) = c_0(1) + c_1(t) + c_2(t^2).$$

Since every $p(t) \in P_2(t)$ can be written this way, $S = \{1, t, t^2\}$ spans $P_2(t)$. $\square$

Example 3.14 — A Set That Does Not Span $\mathbb{R}^3$

Determine whether $S = \{(1, 2, 3), (0, 1, 4)\}$ spans $\mathbb{R}^3$.

Solution: Test whether the vector $(0, 0, 1)$ can be written as a linear combination of S:

$$a(1, 2, 3) + b(0, 1, 4) = (0, 0, 1).$$

This gives:

$$\begin{aligned} a &= 0 \\ 2a + b &= 0 \implies b = 0 \\ 3a + 4b &= 1 \end{aligned}$$

Substituting $a = 0$, $b = 0$ into the third equation gives $0 = 1$, which is a contradiction. Hence $(0, 0, 1) \notin \operatorname{span}(S)$, so S does **not** span $\mathbb{R}^3$.

Key Facts About Spanning Sets:

1. A set of fewer than n vectors can never span $\mathbb{R}^n$.
2. If a system derived from $\operatorname{span}(S) = v$ leads to a contradiction (like $0 = 1$), then $v \notin \operatorname{span}(S)$.
3. A spanning set need not be unique, and it need not be the smallest possible set (it may contain redundant vectors).

Subspaces

Subspace

Let V be a vector space over a field K. A subset $W \subseteq V$ is called a **subspace** of V if W is itself a vector space over K under the addition and scalar multiplication operations already defined on V.

Subspace Test

A subset W of a vector space V is a subspace of V if and only if the following three conditions hold:

1. $0 \in W$ *(zero vector condition)*
2. $u, v \in W \implies u + v \in W$ *(closure under addition)*
3. $u \in W,\ k \in K \implies ku \in W$ *(closure under scalar multiplication)*

Example 3.15 — Past Paper 2023 Q1(vi)

Check whether $W = \{(x, y, z) \in \mathbb{R}^3 : 2x + 3y - 4z = 0\}$ is a subspace of $\mathbb{R}^3$.

Solution: Zero vector: $2(0) + 3(0) - 4(0) = 0$, so $0 = (0, 0, 0) \in W$. ✓

Closure under addition: Let $u = (x_1, y_1, z_1)$ and $v = (x_2, y_2, z_2)$ be in W, so $2x_1 + 3y_1 - 4z_1 = 0$ and $2x_2 + 3y_2 - 4z_2 = 0$. Then

$$\begin{aligned} 2(x_1 + x_2) + 3(y_1 + y_2) - 4(z_1 + z_2) &= (2x_1 + 3y_1 - 4z_1) + (2x_2 + 3y_2 - 4z_2) \\ &= 0 + 0 = 0. \end{aligned}$$

So $u + v \in W$. ✓

Closure under scalar multiplication: For $k \in \mathbb{R}$,

$$2(kx_1) + 3(ky_1) - 4(kz_1) = k(2x_1 + 3y_1 - 4z_1) = k(0) = 0.$$

So $ku \in W$. ✓

All three conditions hold, so W **is** a subspace of $\mathbb{R}^3$. □

Example 3.16 — Past Paper 2021 Q1(vi)

Show that $W = \{(x, y, z) \in \mathbb{R}^3 : 2x + 3y - 4z = 2\}$ is not a subspace of $\mathbb{R}^3$.

Solution: Check whether the zero vector belongs to W:

$$2(0) + 3(0) - 4(0) = 0 \neq 2.$$

Since $0 = (0, 0, 0) \notin W$, the zero vector condition fails. Therefore W is **not** a subspace of $\mathbb{R}^3$. $\square$

Example 3.17 — Past Paper 2024 Q5(a)

Let W be the set of all matrices of the form

$$\begin{bmatrix} a & 0 & b \\ 0 & c & 0 \\ d & 0 & e \end{bmatrix}.$$

Show that W is a subspace of $M_{3,3}$.

Solution: **Zero vector:** Taking $a = b = c = d = e = 0$ gives the zero matrix, which has the required form. So $0 \in W$. ✓

Closure under addition: Let $A = \begin{bmatrix} a_1 & 0 & b_1 \\ 0 & c_1 & 0 \\ d_1 & 0 & e_1 \end{bmatrix}$ and $B = \begin{bmatrix} a_2 & 0 & b_2 \\ 0 & c_2 & 0 \\ d_2 & 0 & e_2 \end{bmatrix}$ be in W. Then

$$A + B = \begin{bmatrix} a_1 + a_2 & 0 & b_1 + b_2 \\ 0 & c_1 + c_2 & 0 \\ d_1 + d_2 & 0 & e_1 + e_2 \end{bmatrix},$$

which has the same form (zero entries in the same fixed positions). So $A + B \in W$. ✓

Closure under scalar multiplication: For $k \in \mathbb{R}$,

$$kA = \begin{bmatrix} ka_1 & 0 & kb_1 \\ 0 & kc_1 & 0 \\ kd_1 & 0 & ke_1 \end{bmatrix},$$

which also has the same form. So $kA \in W$. ✓

All three conditions hold, so W **is** a subspace of $M_{3,3}$. $\square$

Example 3.18 — Closure Fails Under Addition

Determine whether $W = \{(x, y) \in \mathbb{R}^2 : xy = 0\}$ is a subspace of $\mathbb{R}^2$.

Solution: **Zero vector:** $(0)(0) = 0$, so $0 = (0, 0) \in W$. ✓

Closure under addition: Consider $u = (1, 0)$ and $v = (0, 1)$. Both satisfy $xy = 0$, so $u, v \in W$. However,

$$u + v = (1, 1), \qquad (1)(1) = 1 \neq 0.$$

So $u + v \notin W$, and closure under addition **fails**.

Since one condition of the subspace test fails, W is **not** a subspace of $\mathbb{R}^2$, even though the zero vector belongs to it. $\square$

Properties of Subspaces

1. $\{0\}$ is a subspace of every vector space V *(zero subspace)*
2. V is a subspace of itself *(trivial subspace)*
3. If W_1 and W_2 are subspaces of V, then $W_1 \cap W_2$ is also a subspace of V *(intersection property)*
4. The union $W_1 \cup W_2$ is **not** necessarily a subspace of V *(union property)*

Subspace Test — Exam Strategy:

1. Always check the zero vector **first**. If $0 \notin W$, stop immediately and conclude W is not a subspace.
2. A defining equation set equal to a **nonzero** constant (e.g. $= 2$ instead of $= 0$) automatically fails the zero vector test.
3. Watch for nonlinear conditions (e.g. $xy = 0$, $x^2 + y^2 = 1$) or inequalities (e.g. $x \geq 0$) — these usually violate closure under addition or scalar multiplication.
4. Combined test: W is a subspace iff $0 \in W$ and, for all $u, v \in W$ and $a, b \in K$, $au + bv \in W$.

Linear Spans and Row Space of a Matrix

Equation of a Linear Span

Equation of a Linear Span

Given vectors $v_1, v_2, \ldots, v_m$ in K^n, the subspace $W = \operatorname{span}(v_1, \ldots, v_m)$ can often be described by one or more linear equations in $x_1, \ldots, x_n$ that every vector of W must satisfy. To find such an equation: set $a_1v_1 + a_2v_2 + \cdots + a_mv_m = (x_1, \ldots, x_n)$, then eliminate the scalars a_i by row reduction. Any row that reduces to all zeros on the left-hand side yields a linear equation relating $x_1, \ldots, x_n$ — this is the required equation of W.

Example 3.19 — Past Paper 2021 Q4

Find an equation of the subspace W of $\mathbb{R}^3$ spanned by $\{(1, -2, 1), (-2, 0, 3), (3, -2, -2)\}$.

Solution: Set $a(1, -2, 1) + b(-2, 0, 3) + c(3, -2, -2) = (x, y, z)$, giving the coefficient matrix (vectors as columns) augmented by (x, y, z):

$$\left[\begin{array}{ccc|c} 1 & -2 & 3 & x \\ -2 & 0 & -2 & y \\ 1 & 3 & -2 & z \end{array}\right]$$

$R_2 \leftarrow R_2 + 2R_1$, $R_3 \leftarrow R_3 - R_1$:

$$\left[\begin{array}{ccc|c} 1 & -2 & 3 & x \\ 0 & -4 & 4 & y + 2x \\ 0 & 5 & -5 & z - x \end{array}\right]$$

$R_3 \leftarrow R_3 + \frac{5}{4}R_2$:

$$\left[\begin{array}{ccc|c} 1 & -2 & 3 & x \\ 0 & -4 & 4 & y+2x \\ 0 & 0 & 0 & z-x+\frac{5}{4}(y+2x) \end{array}\right]$$

The last row must equal zero for consistency:

$$z - x + \tfrac{5}{4}y + \tfrac{5}{2}x = 0 \implies \tfrac{3}{2}x + \tfrac{5}{4}y + z = 0.$$

Multiplying through by 4:

$$6x + 5y + 4z = 0$$

Verification: for $(1, -2, 1)$: $6(1) + 5(-2) + 4(1) = 6 - 10 + 4 = 0$. ✓

$$W : \boxed{6x + 5y + 4z = 0}$$

Example 3.20 — Past Paper 2023 Q4

Find an equation of the subspace W of $\mathbb{R}^3$ spanned by $(1, -3, 2)$ and $(-2, 0, 3)$.

Solution: Set $a(1, -3, 2) + b(-2, 0, 3) = (x, y, z)$:

$$\left[\begin{array}{cc|c} 1 & -2 & x \\ -3 & 0 & y \\ 2 & 3 & z \end{array}\right]$$

$R_2 \leftarrow R_2 + 3R_1$, $R_3 \leftarrow R_3 - 2R_1$:

$$\left[\begin{array}{cc|c} 1 & -2 & x \\ 0 & -6 & y+3x \\ 0 & 7 & z-2x \end{array}\right]$$

$R_3 \leftarrow R_3 + \frac{7}{6}R_2$:

$$\left[\begin{array}{cc|c} 1 & -2 & x \\ 0 & -6 & y+3x \\ 0 & 0 & z-2x+\frac{7}{6}(y+3x) \end{array}\right]$$

Setting the last row to zero:

$$z - 2x + \tfrac{7}{6}y + \tfrac{7}{2}x = 0 \implies \tfrac{3}{2}x + \tfrac{7}{6}y + z = 0.$$

Multiplying through by 6:

$$9x + 7y + 6z = 0$$

$$W : \boxed{9x + 7y + 6z = 0}$$

Example 3.21 — Equation of a Span in $\mathbb{R}^3$

Find an equation of the subspace spanned by $(1, 1, 0)$ and $(0, 1, 1)$.

Solution: Set $a(1, 1, 0) + b(0, 1, 1) = (x, y, z)$:

$$\left[\begin{array}{cc|c} 1 & 0 & x \\ 1 & 1 & y \\ 0 & 1 & z \end{array}\right]$$

$R_2 \leftarrow R_2 - R_1$:

$$\left[\begin{array}{cc|c} 1 & 0 & x \\ 0 & 1 & y-x \\ 0 & 1 & z \end{array}\right]$$

$R_3 \leftarrow R_3 - R_2$:

$$\left[\begin{array}{cc|c} 1 & 0 & x \\ 0 & 1 & y-x \\ 0 & 0 & z-(y-x) \end{array}\right]$$

Setting the last row to zero: $z - y + x = 0$.

$$W : \boxed{x - y + z = 0}$$

Finding the Equation of a Linear Span:

1. Eliminate the scalars using row reduction; a row of zeros on the left produces the required equation on the right.
2. In $\mathbb{R}^3$, a quick shortcut for two spanning vectors v_1, v_2 is the cross product: the plane through the origin with normal $n = v_1 \times v_2$ has equation $n_1x + n_2y + n_3z = 0$.
3. If the number of independent spanning vectors equals the dimension of the ambient space, the span is the entire space and no restricting equation exists.

Row Space of a Matrix

Row Space of a Matrix

Let A be an $m \times n$ matrix. The **row space** of A, denoted $\text{rowsp}(A)$, is the subspace of K^n spanned by the rows of A. If A and B are row equivalent (one is obtained from the other by elementary row operations), then $\text{rowsp}(A) = \text{rowsp}(B)$. The nonzero rows of any echelon form of A form a basis for $\text{rowsp}(A)$.

Example 3.22 — Basis for a Row Space

Find a basis for the row space of $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 1 & 1 & 1 \end{bmatrix}$.

Solution: Row reduce A: $R_2 \leftarrow R_2 - 2R_1$, $R_3 \leftarrow R_3 - R_1$:

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & -1 & -2 \end{bmatrix}$$

Interchange R_2 and R_3 to reach echelon form:

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & -2 \\ 0 & 0 & 0 \end{bmatrix}$$

The nonzero rows form a basis for $\text{rowsp}(A)$:

$$\boxed{\{(1,2,3),\ (0,-1,-2)\}}, \qquad \dim(\text{rowsp}(A)) = 2$$

Example 3.23 — Basis for a Row Space of a Larger Matrix

Find a basis for the row space of $A = \begin{bmatrix} 1 & 0 & 1 & 2 \\ 2 & 1 & 3 & 4 \\ 0 & 1 & 1 & 0 \end{bmatrix}$.

Solution: $R_2 \leftarrow R_2 - 2R_1$:

$$\begin{bmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

$R_3 \leftarrow R_3 - R_2$:

$$\begin{bmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The nonzero rows form a basis:

$$\boxed{\{(1,0,1,2),\ (0,1,1,0)\}}, \qquad \dim(\text{rowsp}(A)) = 2$$

Example 3.24 — Testing Membership in a Row Space

Determine whether $v = (4,1,5)$ belongs to the row space of $A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 3 & 2 \end{bmatrix}$.

Solution: Seek a, b such that $a(1,2,1) + b(2,3,2) = (4,1,5)$:

$$a + 2b = 4 \tag{1}$$
$$2a + 3b = 1 \tag{2}$$
$$a + 2b = 5 \tag{3}$$

Equations (1) and (3) both equal $a+2b$, but they require $a+2b = 4$ and $a+2b = 5$ simultaneously — a direct contradiction. Hence no such a, b exist.

$$v = (4,1,5)\ \boxed{\text{does NOT belong to rowsp}(A)}$$

Linear Dependence and Independence

Linear Dependence and Independence

Vectors $v_1, v_2, \ldots, v_m$ in a vector space V are said to be **linearly dependent** if there exist scalars $a_1, a_2, \ldots, a_m$, **not all zero**, such that

$$a_1 v_1 + a_2 v_2 + \cdots + a_m v_m = 0.$$

If the only solution is the **trivial** one, $a_1 = a_2 = \cdots = a_m = 0$, then $v_1, \ldots, v_m$ are called **linearly independent**.

Example 3.25 — Testing for Linear Dependence

Determine whether $u_1 = (1,1,2)$, $u_2 = (2,-1,1)$, $u_3 = (4,1,5)$ are linearly dependent in $\mathbb{R}^3$.

Solution: Test whether u_3 is a linear combination of u_1, u_2: set $au_1 + bu_2 = u_3$:

$$a + 2b = 4$$
$$a - b = 1$$
$$2a + b = 5$$

From the second equation, $a = 1 + b$. Substitute into the first: $(1 + b) + 2b = 4 \implies 3b = 3 \implies b = 1$, so $a = 2$.

Verification (third equation): $2(2) + 1(1) = 5$. ✓

Since $u_3 = 2u_1 + u_2$, we have $2u_1 + u_2 - u_3 = 0$ with not all coefficients zero. Hence u_1, u_2, u_3 are **linearly dependent**. □

Example 3.26 — Past Paper 2024 Q4(b)

Determine whether the vectors in $\mathbb{R}^4$ $(1, 3, -1, -4)$, $(3, 8, -5, 7)$, $(2, 9, 4, 23)$ are linearly independent or linearly dependent.

Solution: Set $a(1, 3, -1, -4) + b(3, 8, -5, 7) + c(2, 9, 4, 23) = 0$:

$$a + 3b + 2c = 0 \tag{1}$$
$$3a + 8b + 9c = 0 \tag{2}$$
$$-a - 5b + 4c = 0 \tag{3}$$
$$-4a + 7b + 23c = 0 \tag{4}$$

From (1): $a = -3b - 2c$. Substitute into (2):

$$3(-3b - 2c) + 8b + 9c = -b + 3c = 0 \implies b = 3c.$$

Then $a = -3(3c) - 2c = -11c$.

Check (3): $-(-11c) - 5(3c) + 4c = 11c - 15c + 4c = 0$. ✓(automatically satisfied)

Check (4): $-4(-11c) + 7(3c) + 23c = 44c + 21c + 23c = 88c$.

Setting (4)$= 0$: $88c = 0 \implies c = 0$, which forces $b = 0$ and $a = 0$.

Since only the trivial solution exists, the vectors are **linearly independent**. □

Properties of Linear Dependence and Independence

1. Any set containing the zero vector is linearly dependent *(zero-vector rule)*
2. A single nonzero vector is always linearly independent *(singleton rule)*
3. Two vectors are linearly dependent if and only if one is a scalar multiple of the other *(pair test)*
4. Every subset of a linearly independent set is linearly independent *(hereditary property)*
5. Every superset of a linearly dependent set is linearly dependent *(hereditary property)*
6. Any set of more than n vectors in K^n is automatically linearly dependent *(dimension bound)*

Example 3.27 — Past Paper 2021 Q5

Show that the vectors $(1 - i,\, i)$ and $(2,\, -1 + i)$ in $\mathbb{C}^2$ are linearly dependent over $\mathbb{C}$ but linearly independent over $\mathbb{R}$.

Solution: Over $\mathbb{C}$: Look for a complex scalar k such that $k(1-i,i)=(2,-1+i)$. From the first component:

$$k=\frac{2}{1-i}=\frac{2(1+i)}{(1-i)(1+i)}=\frac{2(1+i)}{2}=1+i.$$

Check the second component: $(1+i)(i)=i+i^2=i-1=-1+i$. ✓

Since $(2,-1+i)=(1+i)(1-i,i)$, the vectors are **linearly dependent over** $\mathbb{C}$.

Over $\mathbb{R}$: Now require $a,b\in\mathbb{R}$ with $a(1-i,i)+b(2,-1+i)=(0,0)$:

$$\text{Component 1: } a(1-i)+2b=(a+2b)-ai=0$$

Separating real and imaginary parts: $a+2b=0$ and $-a=0$, so $a=0$, which forces $b=0$.

Since only the trivial real solution exists, the vectors are **linearly independent over** $\mathbb{R}$.

□

Example 3.28 — Past Paper 2025 Q2(b)

Show that the set $\{1,i\}$ in $\mathbb{C}$ is linearly independent over $\mathbb{R}$ but linearly dependent over $\mathbb{C}$.

Solution: Over $\mathbb{C}$: The complex scalar $k=i$ satisfies $i=k\cdot 1$. Since i is a nonzero complex multiple of 1, the set $\{1,i\}$ is **linearly dependent over** $\mathbb{C}$.

Over $\mathbb{R}$: Suppose $a(1)+b(i)=0$ for real a,b. Then $a+bi=0$. Since a,b are real, this forces both the real part $a=0$ and the imaginary part $b=0$.

Since only the trivial solution exists over $\mathbb{R}$, the set $\{1,i\}$ is **linearly independent over** $\mathbb{R}$. □

Wronskian Test for Functions: For functions $f_1,\dots,f_n$ that are $(n-1)$-times differentiable, the **Wronskian** is

$$W(f_1,\dots,f_n)(x)=\det\begin{bmatrix} f_1 & f_2 & \cdots & f_n \\ f_1' & f_2' & \cdots & f_n' \\ \vdots & \vdots & & \vdots \\ f_1^{(n-1)} & f_2^{(n-1)} & \cdots & f_n^{(n-1)} \end{bmatrix}.$$

If $W(x)$ is **not** identically zero (nonzero for at least one x), then $f_1,\dots,f_n$ are linearly independent.

Example 3.29 — Past Paper 2024 Q1(v) & 2025 Q1(iv)

Use the Wronskian to show that $f_1=1$, $f_2=e^x$, $f_3=e^{2x}$ are linearly independent.

Solution:

$$W(x)=\det\begin{bmatrix} 1 & e^x & e^{2x} \\ 0 & e^x & 2e^{2x} \\ 0 & e^x & 4e^{2x} \end{bmatrix}$$

Expanding along the first column:

$$\begin{aligned} W(x)&=1\cdot\det\begin{bmatrix} e^x & 2e^{2x} \\ e^x & 4e^{2x} \end{bmatrix} \\ &=1\cdot\left(e^x\cdot 4e^{2x}-2e^{2x}\cdot e^x\right) \\ &=4e^{3x}-2e^{3x}=2e^{3x}. \end{aligned}$$

Since $W(x)=2e^{3x}\neq 0$ for every real x, by the Wronskian test f_1,f_2,f_3 are **linearly independent**. □

Example 3.30 — Past Paper 2023 Q1(x)

Let V be the real vector space of functions defined on $\mathbb{R}$. Check whether x and $\cos x$ are linearly dependent or linearly independent.

Solution: Two functions are linearly dependent if and only if one is a scalar multiple of the other for **all** x. Suppose $x = k\cos x$ for some constant k, for all $x \in \mathbb{R}$.

At $x = 0$: $0 = k\cos(0) = k(1) \implies k = 0$.

But if $k = 0$, then $x = 0$ would have to hold for all x, which is false (for example, at $x = 1$, $1 \neq 0$).

Since no single constant k works for all x, the functions x and $\cos x$ are **linearly independent**. □

Quick Tests for Independence:

1. Two vectors: dependent iff one is a scalar multiple of the other.
2. n vectors in K^n: form a matrix with the vectors as rows; independent iff the determinant is nonzero (iff row reduction gives n nonzero rows).
3. Functions: use the Wronskian, or directly test whether one function is a constant multiple of another for all x.
4. Vectors over $\mathbb{C}$: always check dependence over $\mathbb{C}$ **and** over $\mathbb{R}$ separately — the answers may differ.

Basis and Dimension

Basis and Dimension

A set $S = \{u_1, u_2, \ldots, u_n\}$ of vectors in a vector space V is called a **basis** of V if:

1. S is linearly independent, and
2. S spans V.

Every finite-dimensional vector space has a basis, and every basis of V contains the same number of vectors. This number is called the **dimension** of V, denoted $\dim(V)$. The set $\{e_1, e_2, \ldots, e_n\}$, where e_i has 1 in the i-th position and 0 elsewhere, is called the **standard basis** of K^n, and $\dim(K^n) = n$.

Example 3.31 — Past Paper 2021 Q1(viii)

Write the standard basis of $\mathbb{R}^4$.

Solution: The standard basis of $\mathbb{R}^4$ consists of the four unit vectors:

$$e_1 = (1,0,0,0),\ e_2 = (0,1,0,0),\ e_3 = (0,0,1,0),\ e_4 = (0,0,0,1).$$

$$\boxed{\{e_1, e_2, e_3, e_4\}}, \qquad \dim(\mathbb{R}^4) = 4$$

Example 3.32 — Past Paper 2023 Q6 (First Part)

Show that $\{(1,0,1), (0,1,1), (0,0,1)\}$ is a basis of $\mathbb{R}^3$.

Solution: Since $\dim(\mathbb{R}^3) = 3$ and we have exactly 3 vectors, it suffices to check linear independence. Form the matrix with these vectors as rows and compute its determinant:

$$\det \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} = 1(1)(1) = 1 \neq 0$$

(the matrix is upper triangular, so the determinant is the product of the diagonal entries).

Since the determinant is nonzero, the vectors are linearly independent. As there are exactly $3 = \dim(\mathbb{R}^3)$ independent vectors, they automatically span $\mathbb{R}^3$ as well. Hence the set **is a basis** of $\mathbb{R}^3$. □

Example 3.33 — A Set That Is Not a Basis

Determine whether $S = \{(1, 2), (2, 4)\}$ is a basis of $\mathbb{R}^2$.

Solution: Notice that $(2, 4) = 2(1, 2)$, so the second vector is a scalar multiple of the first. Hence S is **linearly dependent**.

Since a basis must first be linearly independent, S fails this requirement. Therefore S is **not a basis** of $\mathbb{R}^2$ (geometrically, both vectors lie along the same line through the origin and cannot span the plane). □

Example 3.34 — Past Paper 2025 Q3(a)

Determine whether $(1, 1, 1, 1)$, $(1, 2, 3, 2)$, $(2, 5, 6, 4)$, $(2, 6, 6, 5)$ form a basis of $\mathbb{R}^4$. If not, find the dimension of the subspace they span.

Solution: Form the matrix with these vectors as rows and row reduce:

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 2 \\ 2 & 5 & 6 & 4 \\ 2 & 6 & 6 & 5 \end{bmatrix}$$

$R_2 \leftarrow R_2 - R_1$, $R_3 \leftarrow R_3 - 2R_1$, $R_4 \leftarrow R_4 - 2R_1$:

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 1 \\ 0 & 3 & 4 & 2 \\ 0 & 4 & 4 & 3 \end{bmatrix}$$

$R_3 \leftarrow R_3 - 3R_2$, $R_4 \leftarrow R_4 - 4R_2$:

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & -2 & -1 \\ 0 & 0 & -4 & -1 \end{bmatrix}$$

$R_4 \leftarrow R_4 - 2R_3$:

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & -2 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

All four rows are nonzero, so the matrix has rank 4. Hence all four vectors are linearly independent. Since 4 independent vectors in a 4-dimensional space automatically span it:

$$\text{the set } \boxed{\text{IS a basis of } \mathbb{R}^4}$$

Shortcuts for Determining a Basis:

1. If $\dim(V) = n$ is known, any n linearly independent vectors in V automatically form a basis — spanning need not be checked separately.
2. Equivalently, any n vectors that span an n-dimensional V automatically form a basis.
3. For vectors in K^n: form a matrix with the vectors as rows and row reduce. n nonzero rows (full rank) means the vectors form a basis.

Example 3.35 — Past Paper 2023 Q1(vii)

Find the dimension of the subspace $W = \{(x_1, x_2, x_3, x_4) : x_2 = x_3\}$ of $\mathbb{R}^4$.

Solution: The condition $x_2 = x_3$ leaves x_1, x_2 (with $x_3 = x_2$), and x_4 free. Write a general vector of W as:

$$(x_1, t, t, x_4) = x_1(1, 0, 0, 0) + t(0, 1, 1, 0) + x_4(0, 0, 0, 1).$$

The set $\{(1, 0, 0, 0), (0, 1, 1, 0), (0, 0, 0, 1)\}$ spans W, and these three vectors are clearly linearly independent (each has a nonzero entry in a position where the others are zero). Hence they form a basis of W.

$$\dim(W) = \boxed{3}$$

Dimension Theorem for Subspaces

If U and W are finite-dimensional subspaces of a vector space V, then $U + W$ is also finite-dimensional and:

$$\dim(U + W) = \dim(U) + \dim(W) - \dim(U \cap W).$$

This result is known as **Grassmann's Formula**.

Example 3.36 — Past Paper 2023 Q5

Suppose U and W are distinct four-dimensional subspaces of a vector space V of dimension six. Find the possible dimension of $U \cap W$.

Solution: Since $U + W$ is a subspace of V, we have $\dim(U + W) \le \dim(V) = 6$.

Also, since $U \ne W$ and both have dimension 4, $U + W$ must be strictly larger than U alone (otherwise $W \subseteq U$ would force $W = U$, since both have the same dimension). Hence $\dim(U + W) \ge 5$.

Combining both bounds: $\dim(U + W) \in \{5, 6\}$.

By Grassmann's Formula:

$$\dim(U \cap W) = \dim(U) + \dim(W) - \dim(U + W) = 8 - \dim(U + W).$$

Case 1: If $\dim(U + W) = 5$, then $\dim(U \cap W) = 8 - 5 = 3$.

Case 2: If $\dim(U + W) = 6$, then $\dim(U \cap W) = 8 - 6 = 2$.

$$\dim(U \cap W) \in \boxed{\{2,\ 3\}}$$

□

Application to Matrices and Rank of a Matrix

Rank of a Matrix

Let A be an $m \times n$ matrix. The **rank** of A, denoted $\text{rank}(A)$, is the dimension of the row space of A. Since the row space and column space of any matrix always have the same dimension (row rank equals column rank), rank can equivalently be computed as the number of nonzero rows in any echelon form of A.

Example 3.37 — Past Paper 2021 Q1(xiv)

Find the rank of $A = \begin{bmatrix} 3 & 0 & 2 & 2 \\ -6 & 42 & 24 & 54 \\ 21 & -21 & 0 & -15 \end{bmatrix}$.

Solution: $R_2 \leftarrow R_2 + 2R_1$, $R_3 \leftarrow R_3 - 7R_1$:

$$\begin{bmatrix} 3 & 0 & 2 & 2 \\ 0 & 42 & 28 & 58 \\ 0 & -21 & -14 & -29 \end{bmatrix}$$

$R_3 \leftarrow R_3 + \frac{1}{2}R_2$:

$$\begin{bmatrix} 3 & 0 & 2 & 2 \\ 0 & 42 & 28 & 58 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

There are 2 nonzero rows in the echelon form.

$$\text{rank}(A) = \boxed{2}$$

Example 3.38 — Rank of a Matrix with Dependent Rows

Find the rank of $A = \begin{bmatrix} 1 & -1 & 2 \\ 2 & -2 & 4 \\ 3 & 1 & 0 \end{bmatrix}$.

Solution: Observe that $R_2 = 2R_1$, so R_1 and R_2 are already dependent. $R_2 \leftarrow R_2 - 2R_1$, $R_3 \leftarrow R_3 - 3R_1$:

$$\begin{bmatrix} 1 & -1 & 2 \\ 0 & 0 & 0 \\ 0 & 4 & -6 \end{bmatrix}$$

Interchanging rows to reach echelon form:

$$\begin{bmatrix} 1 & -1 & 2 \\ 0 & 4 & -6 \\ 0 & 0 & 0 \end{bmatrix}$$

There are 2 nonzero rows.

$$\text{rank}(A) = \boxed{2}$$

Properties of Rank

1. $\text{rank}(A) = \text{rank}(A^T)$ *(row rank equals column rank)*
2. $\text{rank}(A) \leq \min(m, n)$ for an $m \times n$ matrix A *(rank bound)*

3. A square $n \times n$ matrix A is invertible if and only if $\text{rank}(A) = n$ *(invertibility criterion)*
4. $\text{rank}(AB) \le \min(\text{rank}(A), \text{rank}(B))$ *(product rank bound)*
5. Elementary row and column operations do not change the rank of a matrix *(invariance property)*

Example 3.39 — Past Paper 2023 Q2

If $A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 3 \end{bmatrix}$, find a matrix B such that $AB = I_2$. Does this mean A is invertible? Explain.

Solution: We seek a 3×2 matrix B with $AB = I_2$. Writing the columns of B as $(b_1, b_2, b_3)^T$ per column and solving:

First column of B: solve $b_{11} + b_{21} + 2b_{31} = 1$ and $b_{11} + 2b_{21} + 3b_{31} = 0$. Choosing $b_{31} = 0$ gives $b_{21} = -1$ and $b_{11} = 2$.

Second column of B: solve $b_{12} + b_{22} + 2b_{32} = 0$ and $b_{12} + 2b_{22} + 3b_{32} = 1$. Choosing $b_{32} = 0$ gives $b_{22} = 1$ and $b_{12} = -1$.

$$B = \begin{bmatrix} 2 & -1 \\ -1 & 1 \\ 0 & 0 \end{bmatrix}$$

Verification:

$$AB = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ -1 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 2-1+0 & -1+1+0 \\ 2-2+0 & -1+2+0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2. \quad \checkmark$$

Does this mean A is invertible? No. Invertibility is defined only for **square** matrices, and requires a matrix B satisfying **both** $AB = I$ and $BA = I$. Here A is 2×3 (not square), so A cannot have a two-sided inverse. The matrix B found above is only a **right inverse** of A; in fact $BA \neq I_3$, since $\text{rank}(BA) \le \text{rank}(A) = 2 < 3$, so BA cannot equal the 3×3 identity. ☐

Sums and Direct Sums

Sum and Direct Sum of Subspaces

Let U and W be subspaces of a vector space V. The **sum** of U and W is

$$U + W = \{u + w : u \in U,\ w \in W\},$$

which is itself a subspace of V. If every vector $v \in V$ can be written **uniquely** as $v = u + w$ with $u \in U$, $w \in W$, then V is called the **direct sum** of U and W, written $V = U \oplus W$. This holds if and only if:

$$V = U + W \qquad \text{and} \qquad U \cap W = \{0\}.$$

Example 3.40 — Verifying a Direct Sum Decomposition

Let $U = \{(a, b, 0) : a, b \in \mathbb{R}\}$ (the xy-plane) and $W = \{(0, 0, c) : c \in \mathbb{R}\}$ (the z-axis) be subspaces of $\mathbb{R}^3$. Show that $\mathbb{R}^3 = U \oplus W$.

Solution: Spanning: Any $(x, y, z) \in \mathbb{R}^3$ can be written as

$$(x, y, z) = (x, y, 0) + (0, 0, z),$$

with $(x, y, 0) \in U$ and $(0, 0, z) \in W$. So $\mathbb{R}^3 = U + W$.

Trivial intersection: If $(a, b, 0) = (0, 0, c)$, then comparing components gives $a = 0$, $b = 0$, $c = 0$. So $U \cap W = \{(0, 0, 0)\}$.

Since $\mathbb{R}^3 = U + W$ and $U \cap W = \{0\}$:

$$\mathbb{R}^3 = \boxed{U \oplus W}$$

□

Example 3.41 — Confirming a Direct Sum via Dimension

Let $U = \text{span}\{(1, 1, 0), (0, 1, 1)\}$ and $W = \text{span}\{(1, 0, 0)\}$ in $\mathbb{R}^3$. Determine whether $\mathbb{R}^3 = U \oplus W$.

Solution: $\dim(U) = 2$ (the two vectors are independent, not proportional) and $\dim(W) = 1$, so $\dim(U) + \dim(W) = 3 = \dim(\mathbb{R}^3)$.

Check $U \cap W$: A vector in W has the form $(t, 0, 0)$. For this to lie in U: $(t, 0, 0) = a(1, 1, 0) + b(0, 1, 1) = (a, a + b, b)$. Comparing components: $a = t$, $b = 0$, and $a + b = 0 \implies a = 0$. Then $t = a = 0$. So $U \cap W = \{(0, 0, 0)\}$.

By the Dimension Theorem, $\dim(U + W) = \dim(U) + \dim(W) - \dim(U \cap W) = 2 + 1 - 0 = 3 = \dim(\mathbb{R}^3)$, so $U + W = \mathbb{R}^3$.

Since $\mathbb{R}^3 = U + W$ and $U \cap W = \{0\}$:

$$\mathbb{R}^3 = \boxed{U \oplus W}$$

□

Example 3.42 — A Sum That Is Not Direct

Let $U = \text{span}\{(1, 0, 0), (0, 1, 0)\}$ (the xy-plane) and $W = \text{span}\{(1, 1, 0), (0, 0, 1)\}$ in $\mathbb{R}^3$. Show that $\mathbb{R}^3 = U + W$ but $\mathbb{R}^3 \neq U \oplus W$.

Solution: Spanning: Since $(1, 0, 0), (0, 1, 0) \in U$ and $(0, 0, 1) \in W$, the combined set spans all of $\mathbb{R}^3$, so $\mathbb{R}^3 = U + W$.

Intersection is nontrivial: The vector $(1, 1, 0)$ is a basis vector of W, so $(1, 1, 0) \in W$. Also, $(1, 1, 0)$ has z-component 0, so $(1, 1, 0) \in U$ (the xy-plane). Hence $(1, 1, 0) \in U \cap W$, and this vector is **nonzero**.

Since $U \cap W \neq \{0\}$, the sum is **not direct**. For instance, the vector $(1, 1, 0)$ can be decomposed as $(1, 1, 0) + (0, 0, 0)$ or as $(0, 0, 0) + (1, 1, 0)$ — two different decompositions with parts in U and W, violating uniqueness.

$$\mathbb{R}^3 = U + W, \qquad \text{but} \qquad \mathbb{R}^3 \neq \boxed{U \oplus W}$$

□

Properties of Direct Sums

1. $V = U \oplus W$ if and only if $V = U + W$ and $U \cap W = \{0\}$ *(defining criterion)*
2. If $V = U \oplus W$, then $\dim(V) = \dim(U) + \dim(W)$ *(dimension additivity)*
3. If $V = U \oplus W$, every $v \in V$ has a **unique** representation $v = u + w$ with $u \in U$, $w \in W$ *(uniqueness property)*

Coordinates

Coordinate Vector

Let $S = \{u_1, u_2, \ldots, u_n\}$ be a basis of a vector space V. Every vector $v \in V$ can be written **uniquely** as

$$v = a_1u_1 + a_2u_2 + \cdots + a_nu_n.$$

The scalars $(a_1, a_2, \ldots, a_n)$ are called the **coordinates** of v relative to the basis S, and the vector $[v]_S = (a_1, a_2, \ldots, a_n)$ is called the **coordinate vector** of v relative to S.

Example 3.43 — Past Paper 2024 Q1(iv)

Let $S = \{u = (1, 2, 1),\ v = (2, 9, 0),\ w = (3, 3, 4)\}$ be a basis for $\mathbb{R}^3$. Find the vector in $\mathbb{R}^3$ whose coordinate vector relative to S is $(-1, 3, 2)$.

Solution: The required vector is the linear combination:

$$\begin{aligned} -1u + 3v + 2w &= -1(1, 2, 1) + 3(2, 9, 0) + 2(3, 3, 4) \\ &= (-1, -2, -1) + (6, 27, 0) + (6, 6, 8) \\ &= (-1 + 6 + 6,\ -2 + 27 + 6,\ -1 + 0 + 8) \\ &= (11,\ 31,\ 7). \end{aligned}$$

$$\boxed{(11,\ 31,\ 7)}$$

Example 3.44 — Finding a Coordinate Vector

Let $S = \{(1, 1), (1, -1)\}$ be a basis of $\mathbb{R}^2$. Find the coordinate vector of $v = (5, 1)$ relative to S.

Solution: Write $v = a(1, 1) + b(1, -1)$:

$$\begin{aligned} a + b &= 5 \\ a - b &= 1 \end{aligned}$$

Adding the equations: $2a = 6 \implies a = 3$, so $b = 5 - 3 = 2$.

Verification: $3(1, 1) + 2(1, -1) = (3, 3) + (2, -2) = (5, 1)$. ✓

$$[v]_S = \boxed{(3, 2)}$$

Example 3.45 — Coordinates in a Polynomial Space

Let $S = \{1, t, t^2\}$ be the standard basis of $P_2(t)$. Find the coordinate vector of $p(t) = 3 - 2t + 5t^2$ relative to S.

Solution: The polynomial is already expressed as a linear combination of the basis elements:

$$p(t) = 3(1) + (-2)(t) + 5(t^2).$$

Reading off the coefficients in order:

$$[p]_S = \boxed{(3,\ -2,\ 5)}$$

Coordinates — Key Facts:

1. The coordinate vector $[v]_S$ depends on **both** the choice of basis S **and** the order in which its vectors are listed.
2. Relative to the **standard basis** of K^n, the coordinate vector of v is simply v itself.
3. Finding $[v]_S$ always reduces to solving the linear system $v = a_1u_1 + \cdots + a_nu_n$ for the scalars a_i.
4. Given $[v]_S$, recovering v is a direct computation: form the linear combination $a_1u_1 + \cdots + a_nu_n$.

Practice Problems

1. Let $V = \mathbb{R}^2$ with the usual addition, but with scalar multiplication defined by $k(a, b) = (k^2 a,\ k^2 b)$. Show that V is not a vector space by identifying a violated axiom.

2. Let $A = \begin{bmatrix} 2 & -1 \\ 3 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 4 \\ -2 & 5 \end{bmatrix}$ in $M_{2,2}$. Verify that $3(A + B) = 3A + 3B$ using these matrices.

3. In $P_3(t)$, let $p(t) = 2t^3 - t + 4$ and $q(t) = -t^3 + 3t^2 + 1$. Compute $2p(t) - 3q(t)$.

4. In $\mathbb{C}^3$, let $u = (2 - i,\ 1,\ 3i)$ and $v = (i,\ -1 + i,\ 2)$. Find $3u - (1 + i)v$.

5. Determine whether $v = (2, -5, 3)$ can be written as a linear combination of $u_1 = (1, -3, 2)$ and $u_2 = (2, -4, -1)$ in $\mathbb{R}^3$.

6. Find the value of k such that $(2, k, -3)$ is a linear combination of $(1, 2, -1)$ and $(3, 1, 4)$ in $\mathbb{R}^3$.

7. Determine whether $S = \{(1, 1, 1), (2, 1, 0), (3, 2, 1)\}$ spans $\mathbb{R}^3$.

8. Show that $\{1, t, t^2, t^3\}$ spans $P_3(t)$, and determine whether $\{1, t^2\}$ also spans $P_3(t)$.

9. Determine whether $W = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 = z\}$ is a subspace of $\mathbb{R}^3$.

10. Let W be the set of all 2×2 symmetric matrices (i.e. $A = A^T$). Show that W is a subspace of $M_{2,2}$.

11. Determine whether $W = \{(x, y, z) \in \mathbb{R}^3 : x - 2y + z = 0 \text{ and } x + y - z = 3\}$ is a subspace of $\mathbb{R}^3$.

12. Find an equation of the subspace of $\mathbb{R}^3$ spanned by $\{(2, -1, 3), (1, 0, -2), (4, -1, -1)\}$.

13. Find a basis for the row space of $A = \begin{bmatrix} 2 & 1 & 3 & 1 \\ 1 & 0 & 1 & 2 \\ 0 & 1 & 1 & -3 \end{bmatrix}$.

14. Determine whether $u_1 = (1, 0, -1, 2)$, $u_2 = (2, 1, 0, -1)$, $u_3 = (0, -1, -2, 5)$ are linearly dependent or independent in $\mathbb{R}^4$.

15. Show that $f_1 = e^x$, $f_2 = xe^x$, $f_3 = x^2 e^x$ are linearly independent using the Wronskian.

16. Determine whether the vectors $(1, 2, 3, 4)$, $(2, 3, 4, 5)$, $(3, 4, 5, 6)$, $(4, 5, 6, 8)$ in $\mathbb{R}^4$ are linearly dependent or independent.

17. Show that $\{1 - i,\ 2 + i\}$ in $\mathbb{C}$ is linearly dependent over $\mathbb{C}$ but linearly independent over $\mathbb{R}$.

18. Determine whether $\{(1, 0, 2), (0, 1, -1), (2, 1, 3)\}$ is a basis of $\mathbb{R}^3$.

19. Find the dimension of the subspace $W = \{(x_1, x_2, x_3, x_4, x_5) \in \mathbb{R}^5 : x_1 = x_3 \text{ and } x_2 + x_4 = 0\}$ of $\mathbb{R}^5$.

20. If U and W are subspaces of $\mathbb{R}^5$ with $\dim(U) = 3$, $\dim(W) = 4$, and $\dim(U \cap W) = 2$, find $\dim(U + W)$.

21. Suppose U and W are distinct three-dimensional subspaces of a vector space V of dimension five. Find the possible dimensions of $U \cap W$.

22. Find the rank of $A = \begin{bmatrix} 1 & 2 & -1 & 3 \\ 2 & 4 & 1 & -2 \\ 3 & 6 & 3 & -7 \end{bmatrix}$.

23. For what value of k does the matrix $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 1 & k & 3 \end{bmatrix}$ have rank 2 instead of rank 1?

24. Let $U = \{(a, b, c) \in \mathbb{R}^3 : a + b + c = 0\}$ and $W = \text{span}\{(1, 1, 1)\}$. Show that $\mathbb{R}^3 = U \oplus W$.

25. Let $S = \{(1, 2, 3), (0, 1, 2), (0, 0, 1)\}$ be a basis of $\mathbb{R}^3$. Find the coordinate vector of $v = (6, 7, 4)$ relative to S.

Chapter 4

Linear Mappings

Mappings and Functions

Mapping

Let A and B be nonempty sets. A **mapping** (or **function**) $F : A \to B$ is a rule that assigns to each element $a \in A$ a **unique** element $F(a) \in B$. The set A is called the **domain** of F, and B is called the **codomain**. For $a \in A$, the element $F(a) \in B$ is called the **image** of a, and a is called a **preimage** of $F(a)$. The set of all images, $\{F(a) : a \in A\} \subseteq B$, is called the **range** of F.

Example 4.1 — Identifying Domain Codomain and Range

Let $F : \mathbb{R} \to \mathbb{R}$ be defined by $F(x) = x^2$. Identify the domain, codomain, and range of F.

Solution: The domain is $\mathbb{R}$ (all real inputs are allowed), and the codomain is $\mathbb{R}$ (as stated in the definition of F).

Since $x^2 \geq 0$ for every real x, and every nonnegative real number y has a preimage ($x = \sqrt{y}$), the range consists of all nonnegative reals:

$$\text{Domain} = \boxed{\mathbb{R}}, \qquad \text{Codomain} = \boxed{\mathbb{R}}, \qquad \text{Range} = \boxed{[0, \infty)}$$

Example 4.2 — A Mapping Between Vector Spaces

Let $F : \mathbb{R}^2 \to \mathbb{R}^3$ be defined by $F(x, y) = (x + y,\ x - y,\ 2x)$. Find $F(3, 1)$.

Solution: Substitute $x = 3$, $y = 1$ into each component:

$$\begin{aligned} x + y &= 3 + 1 = 4 \\ x - y &= 3 - 1 = 2 \\ 2x &= 2(3) = 6 \end{aligned}$$

$$F(3, 1) = \boxed{(4,\ 2,\ 6)}$$

Example 4.3 — Checking Whether a Rule Is a Mapping

Determine whether $F : \mathbb{R}^2 \to \mathbb{R}$ defined by $F(x, y) = \dfrac{1}{x - y}$ is a mapping on all of $\mathbb{R}^2$.

Solution: For F to be a mapping on the full domain $\mathbb{R}^2$, every element $(x, y) \in \mathbb{R}^2$ must produce a unique well-defined output.

Consider $(x, y) = (2, 2)$. Then $F(2, 2) = \dfrac{1}{2-2} = \dfrac{1}{0}$, which is **undefined**.

Since not every element of $\mathbb{R}^2$ has a defined image, F is **not** a mapping on all of $\mathbb{R}^2$; its actual domain is restricted to $\{(x, y) : x \neq y\}$. □

Types of Mappings:

1. **Injective (one-to-one):** $F(a_1) = F(a_2) \implies a_1 = a_2$ — distinct inputs give distinct outputs.
2. **Surjective (onto):** every $b \in B$ has at least one preimage — the range equals the codomain.
3. **Bijective:** both injective and surjective — every element of B has exactly one preimage.
4. **Composition:** if $F : A \to B$ and $G : B \to C$, the composition $G \circ F : A \to C$ is defined by $(G \circ F)(a) = G(F(a))$.

Linear Mappings

Linear Mapping (Linear Transformation)

Let V and U be vector spaces over the same field K. A mapping $F : V \to U$ is called a **linear mapping** (or **linear transformation**) if it satisfies the following two conditions for all $v_1, v_2 \in V$ and all $k \in K$:

1. $F(v_1 + v_2) = F(v_1) + F(v_2)$ *(additivity)*
2. $F(kv_1) = kF(v_1)$ *(homogeneity)*

Equivalently, F is linear if and only if for all $v_1, v_2 \in V$ and $a, b \in K$:

$$F(av_1 + bv_2) = aF(v_1) + bF(v_2).$$

Every linear mapping satisfies $F(0) = 0$.

Example 4.4 — Verifying a Linear Mapping

Show that $F : \mathbb{R}^3 \to \mathbb{R}^2$ defined by $F(x, y, z) = (2x - y,\ y + 3z)$ is a linear mapping.

Solution: Let $u = (x_1, y_1, z_1)$ and $v = (x_2, y_2, z_2)$ be arbitrary vectors in $\mathbb{R}^3$, and let $a, b \in \mathbb{R}$.

Compute $F(au + bv)$:

$$\begin{aligned} au + bv &= (ax_1 + bx_2,\ ay_1 + by_2,\ az_1 + bz_2) \\ F(au + bv) &= \big(2(ax_1 + bx_2) - (ay_1 + by_2),\ (ay_1 + by_2) + 3(az_1 + bz_2)\big) \\ &= \big(a(2x_1 - y_1) + b(2x_2 - y_2),\ a(y_1 + 3z_1) + b(y_2 + 3z_2)\big). \end{aligned}$$

Compute $aF(u) + bF(v)$:

$$\begin{aligned} aF(u) + bF(v) &= a(2x_1 - y_1,\ y_1 + 3z_1) + b(2x_2 - y_2,\ y_2 + 3z_2) \\ &= \big(a(2x_1 - y_1) + b(2x_2 - y_2),\ a(y_1 + 3z_1) + b(y_2 + 3z_2)\big). \end{aligned}$$

Since $F(au+bv) = aF(u)+bF(v)$ for arbitrary u, v, a, b, F is a **linear mapping**. □

Example 4.5 — A Mapping That Is Not Linear

Show that $F : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $F(x, y) = (x^2,\ y)$ is not a linear mapping.

Solution: It suffices to find one instance where homogeneity or additivity fails. Test homogeneity with $k = 2$ and $v = (1, 1)$:

$$F(kv) = F(2, 2) = (2^2,\ 2) = (4, 2).$$

$$kF(v) = 2F(1, 1) = 2(1^2,\ 1) = 2(1, 1) = (2, 2).$$

Since $F(kv) = (4, 2) \neq (2, 2) = kF(v)$, the homogeneity condition fails.

Therefore F is **not** a linear mapping. □

Example 4.6 — The Zero Mapping and Identity Mapping

Show that the zero mapping $F : V \to U$ defined by $F(v) = 0$ for all $v \in V$, and the identity mapping $I : V \to V$ defined by $I(v) = v$, are both linear mappings.

Solution: Zero mapping: For any $v_1, v_2 \in V$ and $a, b \in K$:

$$F(av_1 + bv_2) = 0 = a(0) + b(0) = aF(v_1) + bF(v_2).$$

So F is linear.

Identity mapping: For any $v_1, v_2 \in V$ and $a, b \in K$:

$$I(av_1 + bv_2) = av_1 + bv_2 = aI(v_1) + bI(v_2).$$

So I is linear. □

Example 4.7 — Linear Mapping Defined by a Matrix

Let $A = \begin{bmatrix} 1 & 2 \\ 0 & -1 \\ 3 & 1 \end{bmatrix}$, and define $F : \mathbb{R}^2 \to \mathbb{R}^3$ by $F(v) = Av$ (treating v as a column vector). Find $F(2, -3)$ and confirm F is linear.

Solution:

$$F(2, -3) = \begin{bmatrix} 1 & 2 \\ 0 & -1 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ -3 \end{bmatrix} = \begin{bmatrix} 1(2) + 2(-3) \\ 0(2) + (-1)(-3) \\ 3(2) + 1(-3) \end{bmatrix} = \begin{bmatrix} -4 \\ 3 \\ 3 \end{bmatrix}$$

$$F(2, -3) = \boxed{(-4,\ 3,\ 3)}$$

To confirm linearity: for any vectors $u, v \in \mathbb{R}^2$ and scalars a, b, matrix multiplication satisfies $A(au+bv) = aAu+bAv$ by the distributive and associative properties of matrix algebra. Hence $F(au + bv) = aF(u) + bF(v)$, confirming F is linear. In fact, **every** mapping of the form $F(v) = Av$ for a fixed matrix A is automatically linear. □

Properties of Linear Mappings

1. $F(0) = 0$ for every linear mapping F *(zero preservation)*
2. $F(-v) = -F(v)$ for all $v \in V$ *(negative preservation)*

3. If $F, G : V \to U$ are linear, then $F + G$ and kF are also linear *(closure under operations)*

4. If $F : V \to U$ and $G : U \to W$ are linear, then $G \circ F : V \to W$ is linear *(composition)*

5. F is completely determined by its action on a basis: if $F(u_1), \ldots, F(u_n)$ are known for a basis $\{u_1, \ldots, u_n\}$ of V, then $F(v)$ is known for every $v \in V$ *(basis determination)*

Example 4.8 — Determining F from Its Action on a Basis

Let $F : \mathbb{R}^2 \to \mathbb{R}^2$ be linear with $F(1, 0) = (2, 3)$ and $F(0, 1) = (-1, 4)$. Find $F(5, -2)$.

Solution: Write $(5, -2)$ in terms of the standard basis:

$$(5, -2) = 5(1, 0) + (-2)(0, 1).$$

By linearity:

$$\begin{aligned} F(5, -2) &= 5F(1, 0) + (-2)F(0, 1) \\ &= 5(2, 3) + (-2)(-1, 4) \\ &= (10, 15) + (2, -8) \\ &= (12,\ 7). \end{aligned}$$

$$F(5, -2) = \boxed{(12,\ 7)}$$

Testing Whether a Mapping Is Linear — Exam Strategy:

1. Every component of $F(x_1, \ldots, x_n)$ must be a **linear** expression in $x_1, \ldots, x_n$ (no squares, products of variables, constants added, or other nonlinear terms).

2. A quick necessary check: if $F(0) \neq 0$, then F is **immediately** not linear.

3. Any mapping of the form $F(v) = Av$ for a fixed matrix A is automatically linear.

4. To disprove linearity, one specific counterexample (failing additivity or homogeneity) is sufficient.

Kernel and Image of a Linear Mapping

Kernel and Image

Let $F : V \to U$ be a linear mapping. The **kernel** of F, denoted $\ker(F)$, is the set of all vectors in V that map to the zero vector of U:

$$\ker(F) = \{v \in V : F(v) = 0\}.$$

The **image** of F, denoted $\mathrm{Im}(F)$, is the set of all vectors in U that are images of some vector in V:

$$\mathrm{Im}(F) = \{F(v) : v \in V\}.$$

Both $\ker(F)$ and $\mathrm{Im}(F)$ are subspaces — $\ker(F)$ is a subspace of V, and $\mathrm{Im}(F)$ is a subspace of U. The dimension of $\ker(F)$ is called the **nullity** of F, and the dimension of $\mathrm{Im}(F)$ is called the **rank** of F.

Example 4.9 — Finding the Kernel of a Linear Mapping

Let $F : \mathbb{R}^3 \to \mathbb{R}^2$ be defined by $F(x, y, z) = (x + y + z,\ x - y - z)$. Find $\ker(F)$.

Solution: We need all (x, y, z) such that $F(x, y, z) = (0, 0)$:

$$\begin{aligned} x + y + z &= 0 \\ x - y - z &= 0 \end{aligned}$$

Adding the two equations: $2x = 0 \implies x = 0$. Substituting into the first equation: $y + z = 0 \implies y = -z$.

So every vector in $\ker(F)$ has the form $(0, -z, z) = z(0, -1, 1)$ for $z \in \mathbb{R}$.

$$\ker(F) = \boxed{\operatorname{span}\{(0, -1, 1)\}}, \qquad \operatorname{nullity}(F) = 1$$

Example 4.10 — Finding the Image of a Linear Mapping

Let $F : \mathbb{R}^3 \to \mathbb{R}^3$ be defined by $F(x, y, z) = (x + y,\ y + z,\ x + 2y + z)$. Find $\operatorname{Im}(F)$.

Solution: Write $F(x, y, z)$ as a linear combination of vectors formed from the coefficients of x, y, z:

$$F(x, y, z) = x(1, 0, 1) + y(1, 1, 2) + z(0, 1, 1).$$

So $\operatorname{Im}(F) = \operatorname{span}\{(1, 0, 1), (1, 1, 2), (0, 1, 1)\}$. Check for dependence: is $(1, 1, 2) = a(1, 0, 1) + b(0, 1, 1)$?

$$\begin{aligned} a &= 1 \\ b &= 1 \\ a + b &= 1 + 1 = 2 \quad \checkmark \end{aligned}$$

Since $(1, 1, 2) = (1, 0, 1) + (0, 1, 1)$, the third spanning vector is redundant. A basis for $\operatorname{Im}(F)$ is $\{(1, 0, 1), (0, 1, 1)\}$, which are independent (not proportional).

$$\operatorname{Im}(F) = \boxed{\operatorname{span}\{(1, 0, 1), (0, 1, 1)\}}, \qquad \operatorname{rank}(F) = 2$$

Example 4.11 — Verifying the Kernel Is a Subspace

Let $F : V \to U$ be a linear mapping. Prove that $\ker(F)$ is a subspace of V.

Solution: Zero vector: Since F is linear, $F(0) = 0$, so $0 \in \ker(F)$. ✓

Closure under addition: Let $v_1, v_2 \in \ker(F)$, so $F(v_1) = 0$ and $F(v_2) = 0$. Then

$$F(v_1 + v_2) = F(v_1) + F(v_2) = 0 + 0 = 0,$$

so $v_1 + v_2 \in \ker(F)$. ✓

Closure under scalar multiplication: Let $v \in \ker(F)$ and $k \in K$. Then

$$F(kv) = kF(v) = k(0) = 0,$$

so $kv \in \ker(F)$. ✓

All three subspace conditions hold, so $\ker(F)$ is a subspace of V. $\square$

Rank-Nullity Theorem

Let $F : V \to U$ be a linear mapping, where V is finite-dimensional. Then:

$$\dim(V) = \dim(\ker(F)) + \dim(\operatorname{Im}(F)),$$

that is,

$$\dim(V) = \operatorname{nullity}(F) + \operatorname{rank}(F).$$

Example 4.12 — Applying the Rank-Nullity Theorem

Let $F : \mathbb{R}^4 \to \mathbb{R}^3$ be a linear mapping with $\dim(\ker(F)) = 2$. Find $\operatorname{rank}(F)$.

Solution: By the Rank-Nullity Theorem:

$$\dim(\mathbb{R}^4) = \operatorname{nullity}(F) + \operatorname{rank}(F).$$

$$4 = 2 + \operatorname{rank}(F) \implies \operatorname{rank}(F) = 2.$$

$$\operatorname{rank}(F) = \boxed{2}$$

Example 4.13 — Kernel and Rank-Nullity Verification

Let $F : \mathbb{R}^3 \to \mathbb{R}^2$ be defined by $F(x, y, z) = (2x - y + z,\ x + y - z)$. Find $\ker(F)$, $\operatorname{Im}(F)$, and verify the Rank-Nullity Theorem.

Solution: Kernel: solve $2x - y + z = 0$ and $x + y - z = 0$ simultaneously. Adding: $3x = 0 \implies x = 0$. Then from the second equation, $y = z$. So $\ker(F) = \{(0, z, z)\} = \operatorname{span}\{(0, 1, 1)\}$, giving $\operatorname{nullity}(F) = 1$.

Image: write

$$F(x, y, z) = x(2, 1) + y(-1, 1) + z(1, -1).$$

Since $(1, -1) = -1(-1, 1)$, the third vector is redundant. Also $(2, 1)$ and $(-1, 1)$ are independent (not proportional). So $\operatorname{Im}(F) = \operatorname{span}\{(2, 1), (-1, 1)\}$. Since these two vectors are independent in $\mathbb{R}^2$, they span all of $\mathbb{R}^2$, giving $\operatorname{rank}(F) = 2$.

Verification:

$$\operatorname{nullity}(F) + \operatorname{rank}(F) = 1 + 2 = 3 = \dim(\mathbb{R}^3). \quad \checkmark$$

□

Singular and Nonsingular Linear Mappings

Nonsingular Linear Mapping and Isomorphism

A linear mapping $F : V \to U$ is called **nonsingular** if $\ker(F) = \{0\}$; otherwise, F is called **singular**. A nonsingular linear mapping is always **injective** (one-to-one).

If $F : V \to U$ is linear, **bijective** (both injective and surjective), then F is called an **isomorphism**, and V and U are said to be **isomorphic**, written $V \cong U$. Two finite-dimensional vector spaces over the same field are isomorphic if and only if they have the **same dimension**.

Example 4.14 — Testing for Nonsingularity

Determine whether $F : \mathbb{R}^3 \to \mathbb{R}^3$ defined by $F(x, y, z) = (x + y,\ y + z,\ x + z)$ is nonsingular.

Solution: Find $\ker(F)$ by solving $F(x,y,z) = (0,0,0)$:

$$x + y = 0 \tag{1}$$
$$y + z = 0 \tag{2}$$
$$x + z = 0 \tag{3}$$

From (1): $y = -x$. From (2): $z = -y = x$. Check (3): $x + z = x + x = 2x = 0 \implies x = 0$. Then $y = 0$, $z = 0$.

Since the only solution is $(0,0,0)$, $\ker(F) = \{0\}$. Hence F is **nonsingular**. $\square$

Example 4.15 — Testing for Singularity

Determine whether $F : \mathbb{R}^3 \to \mathbb{R}^2$ defined by $F(x,y,z) = (x+y+z,\ 2x+2y+2z)$ is singular or nonsingular.

Solution: Find $\ker(F)$: solve $x+y+z = 0$ and $2x+2y+2z = 0$. The second equation is exactly twice the first, so they represent the same condition: $x+y+z = 0$.

This single equation has infinitely many solutions, e.g. $(1,-1,0) \neq (0,0,0)$ satisfies $F(1,-1,0) = (0,0)$.

Since $\ker(F) \neq \{0\}$, F is **singular**. $\square$

Example 4.16 — Verifying an Isomorphism

Let $F : \mathbb{R}^2 \to P_1(t)$ be defined by $F(a,b) = a + bt$. Show that F is an isomorphism.

Solution: **Linearity:** for $(a_1,b_1),(a_2,b_2) \in \mathbb{R}^2$ and scalars c_1, c_2:

$$\begin{aligned} F\big(c_1(a_1,b_1) + c_2(a_2,b_2)\big) &= F(c_1a_1 + c_2a_2,\ c_1b_1 + c_2b_2) \\ &= (c_1a_1 + c_2a_2) + (c_1b_1 + c_2b_2)t \\ &= c_1(a_1 + b_1t) + c_2(a_2 + b_2t) \\ &= c_1F(a_1,b_1) + c_2F(a_2,b_2). \end{aligned}$$

So F is linear.

Nonsingular (injective): if $F(a,b) = 0$, then $a + bt = 0$, which forces $a = 0$ and $b = 0$ (a polynomial is zero only when all coefficients are zero). So $\ker(F) = \{(0,0)\}$, and F is nonsingular, hence injective.

Surjective: every polynomial $p(t) = a + bt \in P_1(t)$ is the image of $(a,b) \in \mathbb{R}^2$, since $F(a,b) = a + bt = p(t)$.

Since F is linear, injective, and surjective, F is an **isomorphism**, so $\mathbb{R}^2 \cong P_1(t)$. This agrees with the fact that $\dim(\mathbb{R}^2) = 2 = \dim(P_1(t))$. $\square$

Properties of Nonsingular Mappings and Isomorphisms

1. F is nonsingular if and only if F is injective *(injectivity criterion)*
2. If $\dim(V) = \dim(U)$, then $F : V \to U$ is nonsingular if and only if F is an isomorphism *(equal-dimension shortcut)*
3. If $F : V \to U$ is nonsingular and $\{v_1, \ldots, v_n\}$ is linearly independent in V, then $\{F(v_1), \ldots, F(v_n)\}$ is linearly independent in U *(independence preservation)*
4. Every n-dimensional vector space over K is isomorphic to K^n *(coordinate isomorphism)*

5. The inverse of an isomorphism $F : V \to U$ is also an isomorphism $F^{-1} : U \to V$ *(inverse property)*

Kernel Rank-Nullity — Exam Strategy:

1. To find $\ker(F)$: set $F(v) = 0$ and solve the resulting homogeneous system.
2. To find $\operatorname{Im}(F)$: write $F(v)$ as a linear combination of vectors formed from the coefficients, then reduce the spanning set to a basis.
3. Use $\dim(V) = \operatorname{nullity}(F) + \operatorname{rank}(F)$ to check work or shortcut a computation once one of the two dimensions is known.
4. F nonsingular $\iff$ $\ker(F) = \{0\}$ $\iff$ F injective $\iff$ $\operatorname{nullity}(F) = 0$.

Operations with Linear Mappings

Operations on Linear Mappings

Let $F, G : V \to U$ be linear mappings and k a scalar. The **sum** $F + G$ and **scalar multiple** kF are the mappings $V \to U$ defined by:

$$(F + G)(v) = F(v) + G(v), \qquad (kF)(v) = kF(v) \quad \text{for all } v \in V.$$

If $F : V \to U$ and $G : U \to W$ are linear, the **composition** $G \circ F : V \to W$ is defined by

$$(G \circ F)(v) = G(F(v)) \quad \text{for all } v \in V.$$

The set of all linear mappings from V to U, denoted $\operatorname{Hom}(V, U)$, is itself a vector space under the sum and scalar multiple defined above.

Example 4.17 — Sum and Scalar Multiple of Linear Mappings

Let $F, G : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by $F(x, y) = (x + y,\ x)$ and $G(x, y) = (2x,\ y - x)$. Find $(F + G)(x, y)$ and $(3F)(x, y)$.

Solution:

$$\begin{aligned}(F + G)(x, y) &= F(x, y) + G(x, y) = (x + y,\ x) + (2x,\ y - x)\\ &= (x + y + 2x,\ x + y - x) = (3x + y,\ y).\end{aligned}$$

$$(3F)(x, y) = 3(x + y,\ x) = (3x + 3y,\ 3x)$$

$$(F + G)(x, y) = \boxed{(3x + y,\ y)}, \qquad (3F)(x, y) = \boxed{(3x + 3y,\ 3x)}$$

Example 4.18 — Composition of Linear Mappings

Let $F, G : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by $F(x, y) = (x + y,\ x - y)$ and $G(x, y) = (2x,\ 3y)$. Find $(G \circ F)(x, y)$.

Solution:

$$(G \circ F)(x, y) = G\big(F(x, y)\big) = G(x + y,\ x - y).$$

Substitute into G, replacing its first argument by $x + y$ and its second by $x - y$:

$$G(x + y,\ x - y) = \big(2(x + y),\ 3(x - y)\big) = (2x + 2y,\ 3x - 3y).$$

$$(G \circ F)(x, y) = \boxed{(2x + 2y,\ 3x - 3y)}$$

Example 4.19 — Proving F+G Is Linear

Let $F, G : V \to U$ be linear mappings. Prove that $F + G$ is also a linear mapping.

Solution: Let $v_1, v_2 \in V$ and $a, b \in K$. By definition of $F + G$ and the linearity of F and G separately:

$$\begin{aligned}(F + G)(av_1 + bv_2) &= F(av_1 + bv_2) + G(av_1 + bv_2) \\ &= \big(aF(v_1) + bF(v_2)\big) + \big(aG(v_1) + bG(v_2)\big) \\ &= a\big(F(v_1) + G(v_1)\big) + b\big(F(v_2) + G(v_2)\big) \\ &= a(F + G)(v_1) + b(F + G)(v_2).\end{aligned}$$

Since $(F + G)(av_1 + bv_2) = a(F + G)(v_1) + b(F + G)(v_2)$ for arbitrary v_1, v_2, a, b, the mapping $F + G$ is linear. □

Properties of the Space of Linear Mappings

1. $\mathrm{Hom}(V, U)$ is a vector space under $+$ and scalar multiplication *(vector space structure)*
2. If $\dim(V) = m$ and $\dim(U) = n$, then $\dim(\mathrm{Hom}(V, U)) = mn$ *(dimension formula)*
3. Composition distributes over addition: $G \circ (F_1 + F_2) = G \circ F_1 + G \circ F_2$ *(distributivity)*
4. Composition is associative: $(H \circ G) \circ F = H \circ (G \circ F)$ *(associativity)*
5. Composition is generally **not** commutative: $G \circ F \neq F \circ G$ in general *(non-commutativity)*

Algebra A(V) of Linear Operators

Linear Operator and the Algebra A(V)

A linear mapping $F : V \to V$ (same domain and codomain) is called a **linear operator** on V. The set of all linear operators on V, denoted $A(V)$, forms an **algebra**: it is a vector space under $+$ and scalar multiplication, and it is also closed under composition, which plays the role of multiplication. For $F, G \in A(V)$, the product FG means $F \circ G$. Powers are defined by $F^2 = F \circ F$, $F^n = F \circ F \circ \cdots \circ F$ (n times), and $F^0 = I$ (the identity operator). For a polynomial $p(t) = a_0 + a_1 t + \cdots + a_n t^n$, the operator

$$p(F) = a_0 I + a_1 F + a_2 F^2 + \cdots + a_n F^n$$

is called a **polynomial in the operator** F. An operator F is **invertible** if there exists $G \in A(V)$ with $FG = GF = I$; this G is denoted F^{-1}.

Example 4.20 — Computing Powers of a Linear Operator

Let $F : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by $F(x, y) = (y,\ -x)$. Find $F^2(x, y)$ and identify the operator F^2.

Solution:

$$\begin{aligned}F^2(x, y) &= F\big(F(x, y)\big) = F(y, -x) \\ &= (-x,\ -y) = -1(x, y).\end{aligned}$$

Since $F^2(x,y) = -(x,y)$ for every (x,y), the operator F^2 is simply -1 times the identity operator:

$$F^2 = \boxed{-I}$$

Example 4.21 — Testing Invertibility of an Operator

Let $F : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by $F(x,y) = (x+y,\ x-y)$. Show that F is invertible and find F^{-1}.

Solution: Nonsingularity: solve $x+y=0$ and $x-y=0$. Adding gives $2x = 0 \implies x = 0$, and then $y = 0$. So $\ker(F) = \{0\}$, hence F is nonsingular. Since F maps $\mathbb{R}^2$ to $\mathbb{R}^2$ (same dimension), F is invertible.

Finding F^{-1}: let $u = x+y$ and $v = x-y$. Adding these: $u+v = 2x \implies x = \dfrac{u+v}{2}$. Subtracting: $u - v = 2y \implies y = \dfrac{u-v}{2}$.

$$F^{-1}(u,v) = \boxed{\left(\frac{u+v}{2},\ \frac{u-v}{2}\right)}$$

Example 4.22 — Evaluating a Polynomial in an Operator

Let $F : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by $F(x,y) = (y,\ x)$ (coordinate swap), and let $p(t) = t^2 - 1$. Compute $p(F)$.

Solution: First compute F^2:

$$F^2(x,y) = F\big(F(x,y)\big) = F(y,x) = (x,y).$$

So $F^2 = I$, the identity operator.

Now evaluate $p(F) = F^2 - I$:

$$p(F)(x,y) = F^2(x,y) - I(x,y) = (x,y) - (x,y) = (0,0).$$

Since $p(F)$ sends every vector to $(0,0)$, $p(F)$ is the **zero operator**:

$$p(F) = \boxed{0}$$

This shows F is a "zero" of the polynomial $p(t) = t^2 - 1$, in the same sense that a matrix can be a zero of a polynomial. □

Properties of the Algebra A(V)

1. $A(V)$ is closed under $+$, scalar multiplication, and composition *(closure)*
2. The identity operator I satisfies $FI = IF = F$ for all $F \in A(V)$ *(multiplicative identity)*
3. $F \in A(V)$ is invertible if and only if F is nonsingular (equivalently, bijective) *(invertibility criterion)*
4. Composition in $A(V)$ is associative but generally **not** commutative: $FG \neq GF$ in general *(non-commutativity)*
5. If F is invertible, then $(F^{-1})^{-1} = F$ and $(FG)^{-1} = G^{-1}F^{-1}$ *(inverse properties)*

Matrices and Linear Mappings

Matrix Representation of a Linear Mapping

Let $F : V \to U$ be a linear mapping, and let $S = \{v_1, \dots, v_n\}$ and $T = \{u_1, \dots, u_m\}$ be bases of V and U respectively. For each v_j, express $F(v_j)$ as a linear combination of the basis T:

$$F(v_j) = a_{1j}u_1 + a_{2j}u_2 + \cdots + a_{mj}u_m.$$

The $m \times n$ matrix $A = [a_{ij}]$, whose j-th column is the coordinate vector $[F(v_j)]_T$, is called the **matrix representation** of F relative to the bases S and T, denoted $[F]_{S,T}$ (or simply $[F]_S$ when $V = U$ and $S = T$). This matrix satisfies the key property:

$$[F(v)]_T = [F]_{S,T}\,[v]_S \qquad \text{for every } v \in V.$$

Example 4.23 — Matrix Representation Relative to Standard Bases

Let $F : \mathbb{R}^3 \to \mathbb{R}^2$ be defined by $F(x, y, z) = (2x - y + z,\ x + 3y - z)$. Find the matrix representation of F relative to the standard bases.

Solution: Apply F to each standard basis vector of $\mathbb{R}^3$:

$$\begin{aligned} F(1,0,0) &= (2,1) \\ F(0,1,0) &= (-1,3) \\ F(0,0,1) &= (1,-1) \end{aligned}$$

Using these images as the columns of the matrix:

$$[F] = \boxed{\begin{bmatrix} 2 & -1 & 1 \\ 1 & 3 & -1 \end{bmatrix}}$$

Example 4.24 — Matrix Representation Relative to a Nonstandard Basis

Let $F : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by $F(x, y) = (x + y,\ x - y)$. Find the matrix representation of F relative to the basis $S = \{(1,1), (1,-1)\}$.

Solution: Image of $(1,1)$**:** $F(1,1) = (1+1,\ 1-1) = (2,0)$. Express $(2,0)$ in terms of S: $a(1,1) + b(1,-1) = (2,0)$ gives $a + b = 2$, $a - b = 0$, so $a = 1$, $b = 1$. Coordinate vector: $(1,1)$.

Image of $(1,-1)$**:** $F(1,-1) = (1-1,\ 1+1) = (0,2)$. Express $(0,2)$ in terms of S: $a+b = 0$, $a - b = 2$, so $a = 1$, $b = -1$. Coordinate vector: $(1,-1)$.

Using these coordinate vectors as columns:

$$[F]_S = \boxed{\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}}$$

Example 4.25 — Using the Matrix Representation to Compute F(v)

Using the mapping and basis S from Example 4.24, let $v = (3, -1)$ with coordinate vector $[v]_S = (1, 2)$. Verify that $[F(v)]_S = [F]_S\,[v]_S$.

Solution: Direct computation: $F(3,-1) = (3 + (-1),\ 3 - (-1)) = (2,4)$. Express $(2,4)$ in terms of S: $a + b = 2$, $a - b = 4$, so $a = 3$, $b = -1$. Thus $[F(v)]_S = (3,-1)$.

Matrix computation:

$$[F]_S\,[v]_S = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1(1) + 1(2) \\ 1(1) - 1(2) \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}.$$

Both methods give $(3, -1)$, confirming $[F(v)]_S = [F]_S\,[v]_S$. ✓ □

Matrix Representation — Key Facts:

1. Relative to the **standard basis**, the matrix representation of $F(v) = Av$ is simply A itself.
2. The columns of $[F]_{S,T}$ are the coordinate vectors (with respect to T) of F applied to each vector of S, in order.
3. Composition corresponds to matrix multiplication: $[G \circ F]_S = [G]_S[F]_S$.
4. F is invertible if and only if $[F]_S$ is an invertible matrix, and then $[F^{-1}]_S = ([F]_S)^{-1}$.

Practice Problems

1. Let $G : \mathbb{R}^3 \to \mathbb{R}^2$ be defined by $G(x, y, z) = (2x - z,\ y + 3z)$. Determine whether G is injective, surjective, both, or neither.

2. Let $H : \mathbb{R} \to \mathbb{R}$ be defined by $H(x) = \dfrac{2}{x^2 - 4}$. Determine the largest possible domain of H as a subset of $\mathbb{R}$.

3. Let $F, G : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by $F(x, y) = (x + 2y,\ 3x - y)$ and $G(x, y) = (y,\ x + y)$. Find $(G \circ F)(1, -2)$ and $(F \circ G)(1, -2)$, and comment on whether $G \circ F = F \circ G$.

4. Determine whether $F : \mathbb{R}^3 \to \mathbb{R}^2$ defined by $F(x, y, z) = (x + 2y - z,\ 3)$ is a linear mapping.

5. Determine whether $F : M_{2,2} \to \mathbb{R}$ defined by $F(A) = \det(A)$ is a linear mapping.

6. Let $F : \mathbb{R}^3 \to \mathbb{R}^3$ be linear with $F(1, 0, 0) = (2, 1, -1)$, $F(0, 1, 0) = (0, 3, 1)$, and $F(0, 0, 1) = (1, -1, 2)$. Find $F(2, -1, 3)$.

7. Let $F : \mathbb{R}^3 \to \mathbb{R}^3$ be defined by $F(x, y, z) = (x - y,\ y - z,\ z - x)$. Find $\ker(F)$ and $\operatorname{Im}(F)$, and verify the Rank-Nullity Theorem.

8. Let $F : \mathbb{R}^4 \to \mathbb{R}^3$ be defined by $F(x_1, x_2, x_3, x_4) = (x_1 + x_2,\ x_2 + x_3,\ x_3 + x_4)$. Find a basis for $\ker(F)$.

9. Let $F : \mathbb{R}^3 \to \mathbb{R}^2$ be a linear mapping with $\operatorname{rank}(F) = 2$. Find $\dim(\ker(F))$.

10. Let $F : P_2(t) \to P_1(t)$ be defined by $F(a_0 + a_1 t + a_2 t^2) = a_1 + 2a_2 t$. Find $\ker(F)$ and $\operatorname{Im}(F)$.

11. Determine whether $F : \mathbb{R}^3 \to \mathbb{R}^3$ defined by $F(x, y, z) = (x + y,\ y + z,\ x + 2y + z)$ is singular or nonsingular.

12. Determine whether $F : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $F(x, y) = (3x - y,\ x + 2y)$ is an isomorphism. If so find F^{-1}.

13. Show that $M_{2,2}$ is isomorphic to $\mathbb{R}^4$ by exhibiting an explicit isomorphism.

14. Let $F, G : \mathbb{R}^3 \to \mathbb{R}^2$ be defined by $F(x, y, z) = (x - z,\ 2y)$ and $G(x, y, z) = (x + y,\ z - y)$. Find $(2F - 3G)(x, y, z)$.

15. Let $F : \mathbb{R}^2 \to \mathbb{R}^3$ and $G : \mathbb{R}^3 \to \mathbb{R}^2$ be defined by $F(x, y) = (x,\ x + y,\ y)$ and $G(x, y, z) = (x - z,\ y)$. Find $(G \circ F)(x, y)$ and determine whether $G \circ F$ is the identity mapping on $\mathbb{R}^2$.

16. Prove that if $F : V \to U$ is a linear mapping and k is a scalar, then kF is also a linear mapping.

17. Let $F : \mathbb{R}^3 \to \mathbb{R}^3$ be defined by $F(x, y, z) = (z,\ x,\ y)$. Find F^2 and F^3, and determine whether F is invertible.

18. Let $F : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by $F(x, y) = (2x,\ 3y)$. Show that F is invertible and find F^{-1}.

19. Let $F : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by $F(x, y) = (-x,\ y)$. Show that F is a zero of the polynomial $p(t) = t^2 - 1$ by computing $p(F) = F^2 - I$.

20. Let F, G be linear operators on $\mathbb{R}^2$ defined by $F(x, y) = (y, 0)$ and $G(x, y) = (0, x)$. Compute FG and GF, and determine whether $FG = GF$.

21. Find the matrix representation of $F : \mathbb{R}^3 \to \mathbb{R}^2$ defined by $F(x, y, z) = (3x - y + 2z,\ x + 4z)$ relative to the standard bases.

22. Let $F : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by $F(x, y) = (2x - y,\ x + y)$. Find the matrix representation of F relative to the basis $S = \{(1, 2), (2, 3)\}$.

23. Let $F : \mathbb{R}^2 \to \mathbb{R}^2$ have matrix representation $[F]_S = \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix}$ relative to a basis $S = \{u_1, u_2\}$. Given that a vector v has coordinate vector $[v]_S = (3, -2)$, find $[F(v)]_S$.

24. Let $F : \mathbb{R}^3 \to \mathbb{R}^3$ be defined by $F(x, y, z) = (x + y + z,\ 2x + ky + 2z,\ x + y + z)$. Find all values of k for which F is singular.

25. Let $F : P_1(t) \to \mathbb{R}^2$ be defined by $F(a + bt) = (a - b,\ a + b)$. Show that F is an isomorphism, and find a formula for $F^{-1}(x, y)$.

Chapter 5

Inner Product Spaces

Inner Product Spaces

Inner Product Space

Let V be a vector space over $\mathbb{R}$ (or $\mathbb{C}$). An **inner product** on V is a function that assigns to every pair of vectors $u, v \in V$ a scalar $\langle u, v\rangle$ satisfying the following axioms for all $u, v, w \in V$ and $k \in \mathbb{R}$ (or $\mathbb{C}$):

1. $\langle u + v,\ w\rangle = \langle u, w\rangle + \langle v, w\rangle$ *(linearity in first slot)*
2. $\langle ku,\ v\rangle = k\langle u, v\rangle$ *(homogeneity)*
3. $\langle u, v\rangle = \overline{\langle v, u\rangle}$ *(conjugate symmetry; $= \langle v, u\rangle$ over $\mathbb{R}$)*
4. $\langle u, u\rangle \geq 0$, and $\langle u, u\rangle = 0$ if and only if $u = 0$ *(positive definiteness)*

A vector space V equipped with an inner product is called an **inner product space**. The **norm** (length) of a vector u is defined as

$$\|u\| = \sqrt{\langle u, u\rangle}.$$

A vector u with $\|u\| = 1$ is called a **unit vector**, and the process of replacing u with $u/\|u\|$ is called **normalization**.

Example 5.1 — The Standard Inner Product on $\mathbb{R}^n$

Let $u = (1, -2, 3)$ and $v = (4, 0, -1)$ in $\mathbb{R}^3$. Using the standard (Euclidean) inner product $\langle u, v\rangle = u_1v_1 + u_2v_2 + u_3v_3$, compute $\langle u, v\rangle$.

Solution:

$$\langle u, v\rangle = (1)(4) + (-2)(0) + (3)(-1) = 4 + 0 - 3 = 1.$$

$$\langle u, v\rangle = \boxed{1}$$

Example 5.2 — Computing a Norm

Find $\|u\|$ for $u = (1, -5, 3)$ in $\mathbb{R}^3$ using the standard inner product.

Solution:

$$\langle u, u\rangle = 1^2 + (-5)^2 + 3^2 = 1 + 25 + 9 = 35.$$

$$\|u\| = \sqrt{35}$$

$$\|u\| = \boxed{\sqrt{35}}$$

Example 5.3 — Past Paper 2024 Q1(xii) & 2025 Q1(x)

Normalize the vector $v = (1, 2, 4, 5)$.

Solution: First compute the norm:

$$\|v\| = \sqrt{1^2 + 2^2 + 4^2 + 5^2} = \sqrt{1 + 4 + 16 + 25} = \sqrt{46}.$$

The normalized (unit) vector is obtained by dividing each component by $\|v\|$:

$$\hat{v} = \frac{v}{\|v\|} = \left(\frac{1}{\sqrt{46}}, \frac{2}{\sqrt{46}}, \frac{4}{\sqrt{46}}, \frac{5}{\sqrt{46}}\right).$$

Verification: $\|\hat{v}\|^2 = \dfrac{1 + 4 + 16 + 25}{46} = \dfrac{46}{46} = 1.$ ✓

$$\hat{v} = \boxed{\left(\frac{1}{\sqrt{46}}, \frac{2}{\sqrt{46}}, \frac{4}{\sqrt{46}}, \frac{5}{\sqrt{46}}\right)}$$

Properties of the Inner Product

1. $\langle u,\ v + w\rangle = \langle u, v\rangle + \langle u, w\rangle$ *(linearity in second slot, over $\mathbb{R}$)*
2. $\langle u,\ kv\rangle = k\langle u, v\rangle$ over $\mathbb{R}$; $\langle u,\ kv\rangle = \overline{k}\langle u, v\rangle$ over $\mathbb{C}$ *(conjugate-homogeneity in second slot)*
3. $\langle 0,\ v\rangle = \langle v,\ 0\rangle = 0$ for every $v \in V$ *(zero vector property)*
4. $\|kv\| = |k|\,\|v\|$ for any scalar k *(norm scaling)*
5. $\|u\| \geq 0$, with $\|u\| = 0$ if and only if $u = 0$ *(positive definiteness of norm)*

Examples of Inner Product Spaces

Weighted Euclidean Inner Product

Weighted Inner Product on $\mathbb{R}^n$

Given fixed positive weights $w_1, w_2, \ldots, w_n$, the function

$$\langle u, v\rangle = w_1u_1v_1 + w_2u_2v_2 + \cdots + w_nu_nv_n$$

defines a valid inner product on $\mathbb{R}^n$, called a **weighted Euclidean inner product**. Choosing all weights equal to 1 recovers the standard (unweighted) inner product.

Example 5.4 — Computing a Weighted Inner Product

Let $u = (1, 2)$ and $v = (3, -1)$ in $\mathbb{R}^2$, with weights $w_1 = 2$ and $w_2 = 5$. Compute $\langle u, v\rangle$ using this weighted inner product.

Solution:

$$\langle u, v\rangle = w_1u_1v_1 + w_2u_2v_2 = 2(1)(3) + 5(2)(-1) = 6 - 10 = -4.$$

$$\langle u, v\rangle = \boxed{-4}$$

Complex Inner Product Space $\mathbb{C}^n$

Standard Inner Product on $\mathbb{C}^n$

For $u = (u_1, \ldots, u_n)$ and $v = (v_1, \ldots, v_n)$ in $\mathbb{C}^n$, the standard inner product is defined using complex conjugates to guarantee $\langle u, u\rangle \geq 0$:

$$\langle u, v\rangle = u_1\overline{v_1} + u_2\overline{v_2} + \cdots + u_n\overline{v_n}.$$

Example 5.5 — Inner Product in $\mathbb{C}^2$

Let $u = (2 + i,\ 1 - i)$ and $v = (1,\ 2 + i)$ in $\mathbb{C}^2$. Compute $\langle u, v\rangle$.

Solution:

$$\begin{aligned}
\langle u, v\rangle &= (2 + i)\overline{(1)} + (1 - i)\overline{(2 + i)} \\
&= (2 + i)(1) + (1 - i)(2 - i) \\
&= (2 + i) + \left(2 - i - 2i + i^2\right) \\
&= (2 + i) + (2 - 3i - 1) \\
&= (2 + i) + (1 - 3i) \\
&= 3 - 2i.
\end{aligned}$$

$$\langle u, v\rangle = \boxed{3 - 2i}$$

Function Space Inner Product

Inner Product on a Function Space

Let $V = C[a, b]$ be the vector space of continuous real functions on the interval $[a, b]$. The function

$$\langle f, g\rangle = \int_a^b f(t)g(t)\,dt$$

defines an inner product on $C[a, b]$.

Example 5.6 — Inner Product of Functions

Let $f(t) = t$ and $g(t) = t^2$ in $C[0, 1]$. Compute $\langle f, g\rangle$ using $\langle f, g\rangle = \int_0^1 f(t)g(t)\,dt$.

Solution:

$$\langle f, g\rangle = \int_0^1 t \cdot t^2\,dt = \int_0^1 t^3\,dt = \left[\frac{t^4}{4}\right]_0^1 = \frac{1}{4} - 0 = \frac{1}{4}.$$

$$\langle f, g\rangle = \boxed{\frac{1}{4}}$$

Matrix Space Inner Product

Inner Product on $M_{m,n}$

For matrices $A, B \in M_{m,n}(\mathbb{R})$, the function

$$\langle A, B\rangle = \operatorname{tr}(B^T A)$$

defines an inner product on $M_{m,n}$, where tr denotes the trace (sum of diagonal entries). This is equivalent to summing the products of corresponding entries of A and B.

Example 5.7 — Inner Product of Matrices

Let $A = \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 1 \\ 3 & 4 \end{bmatrix}$ in $M_{2,2}$. Compute $\langle A, B \rangle$.

Solution: Sum the products of corresponding entries:

$$\langle A, B \rangle = (1)(2) + (2)(1) + (0)(3) + (-1)(4) = 2 + 2 + 0 - 4 = 0.$$

$$\langle A, B \rangle = \boxed{0}$$

Common Inner Product Spaces to Remember:

1. $\mathbb{R}^n$ with the standard (Euclidean) inner product $\langle u, v \rangle = \sum u_i v_i$
2. $\mathbb{R}^n$ with a weighted inner product $\langle u, v \rangle = \sum w_i u_i v_i$, $w_i > 0$
3. $\mathbb{C}^n$ with $\langle u, v \rangle = \sum u_i \overline{v_i}$ (conjugate on the **second** vector)
4. $C[a, b]$ with $\langle f, g \rangle = \int_a^b f(t)g(t)\,dt$
5. $M_{m,n}$ with $\langle A, B \rangle = \operatorname{tr}(B^T A)$ (sum of products of corresponding entries)
6. When $\langle u, v \rangle = 0$ for two nonzero vectors, they are called **orthogonal** (covered in Section 5.5).

Cauchy-Schwarz Inequality and Applications

Cauchy-Schwarz Inequality

Let u, v be vectors in an inner product space V. Then:

$$|\langle u, v \rangle| \leq \|u\|\,\|v\|.$$

Equality holds if and only if u and v are linearly dependent (one is a scalar multiple of the other).

Triangle Inequality

Let u, v be vectors in an inner product space V. Then:

$$\|u + v\| \leq \|u\| + \|v\|.$$

This is also called **Minkowski's Inequality** and follows directly from the Cauchy-Schwarz Inequality.

Example 5.8 — Past Paper 2023 Q1(i)

If V is an inner product space and $u, v \in V$, show that

$$\cos\theta = \frac{\langle u, v \rangle}{\|u\|\,\|v\|}.$$

Solution: By the Cauchy-Schwarz Inequality, $|\langle u, v \rangle| \leq \|u\|\,\|v\|$ for all nonzero $u, v \in V$.

Dividing both sides by $\|u\|\,\|v\| > 0$:

$$-1 \leq \frac{\langle u, v\rangle}{\|u\|\,\|v\|} \leq 1.$$

Since the quantity $\dfrac{\langle u, v\rangle}{\|u\|\,\|v\|}$ always lies in $[-1, 1]$, it is a valid value for the cosine of some angle θ with $0 \leq \theta \leq \pi$. This angle is defined to be the **angle between** u **and** v, giving:

$$\cos\theta = \boxed{\frac{\langle u, v\rangle}{\|u\|\,\|v\|}}$$

□

Example 5.9 — Past Paper 2021 Q1(i)

Compute the angle between vectors $u = (-1, 1, 2)$ and $v = (2, 1, -1)$.

Solution:

$$\langle u, v\rangle = (-1)(2) + (1)(1) + (2)(-1) = -2 + 1 - 2 = -3.$$

$$\|u\| = \sqrt{(-1)^2 + 1^2 + 2^2} = \sqrt{6}, \qquad \|v\| = \sqrt{2^2 + 1^2 + (-1)^2} = \sqrt{6}.$$

$$\cos\theta = \frac{\langle u, v\rangle}{\|u\|\,\|v\|} = \frac{-3}{\sqrt{6}\cdot\sqrt{6}} = \frac{-3}{6} = -\frac{1}{2}.$$

$$\theta = \cos^{-1}\left(-\frac{1}{2}\right)$$

$$\theta = \boxed{120^\circ}$$

Example 5.10 — Verifying the Cauchy-Schwarz Inequality

Verify the Cauchy-Schwarz Inequality for $u = (1, 2, 3)$ and $v = (4, -1, 2)$ in $\mathbb{R}^3$.

Solution:

$$\langle u, v\rangle = (1)(4) + (2)(-1) + (3)(2) = 4 - 2 + 6 = 8.$$

$$|\langle u, v\rangle| = 8.$$

$$\|u\| = \sqrt{1 + 4 + 9} = \sqrt{14}, \qquad \|v\| = \sqrt{16 + 1 + 4} = \sqrt{21}.$$

$$\|u\|\,\|v\| = \sqrt{14}\cdot\sqrt{21} = \sqrt{294} \approx 17.15.$$

Since $8 \leq 17.15$:

$$|\langle u, v\rangle| \leq \|u\|\,\|v\| \quad \checkmark$$

The Cauchy-Schwarz Inequality holds. □

Example 5.11 — Verifying the Triangle Inequality

Verify the Triangle Inequality for $u = (3, 0, 4)$ and $v = (1, 2, 2)$ in $\mathbb{R}^3$.

Solution:

$$u + v = (4, 2, 6).$$

$$\|u + v\| = \sqrt{4^2 + 2^2 + 6^2} = \sqrt{16 + 4 + 36} = \sqrt{56} \approx 7.48.$$

$$\|u\| = \sqrt{9 + 0 + 16} = \sqrt{25} = 5, \qquad \|v\| = \sqrt{1 + 4 + 4} = \sqrt{9} = 3.$$

$$\|u\| + \|v\| = 5 + 3 = 8.$$

Since $7.48 \leq 8$:

$$\|u + v\| \leq \|u\| + \|v\| \quad \checkmark$$

The Triangle Inequality holds. $\square$

Example 5.12 — Past Paper 2021 Q1(xii)

Find the norm of $\left(\frac{1}{2}, -\frac{1}{4}, \frac{1}{3}, \frac{1}{6}\right) \in \mathbb{R}^4$ with respect to the Euclidean inner product on $\mathbb{R}^4$.

Solution:

$$\|v\|^2 = \left(\frac{1}{2}\right)^2 + \left(-\frac{1}{4}\right)^2 + \left(\frac{1}{3}\right)^2 + \left(\frac{1}{6}\right)^2 = \frac{1}{4} + \frac{1}{16} + \frac{1}{9} + \frac{1}{36}.$$

Using the common denominator 144:

$$\|v\|^2 = \frac{36}{144} + \frac{9}{144} + \frac{16}{144} + \frac{4}{144} = \frac{65}{144}.$$

$$\|v\| = \sqrt{\frac{65}{144}} = \frac{\sqrt{65}}{12}$$

$$\|v\| = \boxed{\frac{\sqrt{65}}{12}}$$

Cauchy-Schwarz — Key Facts:

1. Cauchy-Schwarz guarantees $\dfrac{\langle u, v\rangle}{\|u\|\|v\|} \in [-1, 1]$, which is what makes the angle formula well-defined.
2. Equality in Cauchy-Schwarz occurs exactly when u, v are parallel (one is a scalar multiple of the other).
3. $\theta = 90^\circ$ (i.e. $\cos\theta = 0$) exactly when $\langle u, v\rangle = 0$ — this is the definition of orthogonality, covered next.
4. Always compute $\langle u, v\rangle$, $\|u\|$, and $\|v\|$ separately before substituting into the cosine formula, to avoid arithmetic errors.

Orthogonality

Orthogonal Vectors and Orthogonal Complement

Vectors u, v in an inner product space V are called **orthogonal** if $\langle u, v\rangle = 0$. For a subset $S \subseteq V$, the **orthogonal complement** of S, denoted $S^\perp$, is the set of all vectors in V orthogonal to every vector in S:

$$S^\perp = \{v \in V : \langle v, s\rangle = 0 \text{ for all } s \in S\}.$$

$S^\perp$ is always a subspace of V, even if S itself is not.

Example 5.13 — Checking Orthogonality

Determine whether $u = (2, -1, 3)$ and $v = (1, 5, 1)$ are orthogonal in $\mathbb{R}^3$.

Solution:

$$\langle u, v\rangle = (2)(1) + (-1)(5) + (3)(1) = 2 - 5 + 3 = 0.$$

Since $\langle u, v\rangle = 0$, the vectors **are** orthogonal. □

Example 5.14 — Past Paper 2023 Q1(xii)

Find a vector orthogonal to $u_1 = \begin{bmatrix}1\\2\\1\end{bmatrix}$ and $u_2 = \begin{bmatrix}2\\1\\-4\end{bmatrix}$.

Solution: A vector orthogonal to both u_1 and u_2 can be found using the cross product $u_1 \times u_2$:

$$u_1 \times u_2 = \begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 1 & 2 & 1\\ 2 & 1 & -4\end{vmatrix}$$

$$\begin{aligned}
&= \mathbf{i}\big((2)(-4) - (1)(1)\big) - \mathbf{j}\big((1)(-4) - (1)(2)\big) + \mathbf{k}\big((1)(1) - (2)(2)\big)\\
&= \mathbf{i}(-8 - 1) - \mathbf{j}(-4 - 2) + \mathbf{k}(1 - 4)\\
&= -9\mathbf{i} + 6\mathbf{j} - 3\mathbf{k}.
\end{aligned}$$

So $w = (-9, 6, -3)$, or dividing by -3, the simplified vector $w = (3, -2, 1)$.

Verification: $\langle u_1, w\rangle = 3 - 4 + 1 = 0.$ ✓ $\langle u_2, w\rangle = 6 - 2 - 4 = 0.$ ✓

$$w = \boxed{(3,\ -2,\ 1)}$$

Example 5.15 — Finding an Orthogonal Complement

Find $W^\perp$ where $W = \operatorname{span}\{(1, 1, 0)\}$ in $\mathbb{R}^3$.

Solution: We need all (x, y, z) such that $\langle (x, y, z), (1, 1, 0)\rangle = 0$:

$$x + y = 0 \implies y = -x, \quad z \text{ free}.$$

So every vector in $W^\perp$ has the form $(x, -x, z) = x(1, -1, 0) + z(0, 0, 1)$.

$$W^\perp = \boxed{\operatorname{span}\{(1, -1, 0),\ (0, 0, 1)\}}, \qquad \dim(W^\perp) = 2$$

Orthogonal Matrices

Orthogonal Matrix

A real square matrix P is called **orthogonal** if

$$P^T P = P P^T = I,$$

equivalently, $P^{-1} = P^T$. This holds if and only if the rows of P (and equivalently the columns of P) form an orthonormal set of vectors.

Example 5.16 — Verifying an Orthogonal Matrix

Show that $P = \begin{bmatrix} \frac{3}{5} & \frac{4}{5} \\ -\frac{4}{5} & \frac{3}{5} \end{bmatrix}$ is an orthogonal matrix.

Solution: Compute P^TP:

$$P^T = \begin{bmatrix} \frac{3}{5} & -\frac{4}{5} \\ \frac{4}{5} & \frac{3}{5} \end{bmatrix}$$

$$P^TP = \begin{bmatrix} \frac{3}{5} & -\frac{4}{5} \\ \frac{4}{5} & \frac{3}{5} \end{bmatrix} \begin{bmatrix} \frac{3}{5} & \frac{4}{5} \\ -\frac{4}{5} & \frac{3}{5} \end{bmatrix} = \begin{bmatrix} \frac{9}{25} + \frac{16}{25} & \frac{12}{25} - \frac{12}{25} \\ \frac{12}{25} - \frac{12}{25} & \frac{16}{25} + \frac{9}{25} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I.$$

Since $P^TP = I$, P **is** an orthogonal matrix. □

Example 5.17 — Past Paper 2021 Q1(xi)

If A is an orthogonal matrix, show that $\det A = 1$ or -1.

Solution: Since A is orthogonal, $A^TA = I$. Taking the determinant of both sides:

$$\det(A^TA) = \det(I) = 1.$$

Using the property $\det(A^TA) = \det(A^T)\det(A)$ and $\det(A^T) = \det(A)$:

$$\det(A) \cdot \det(A) = 1 \implies (\det A)^2 = 1.$$

Taking the square root of both sides:

$$\det A = \pm 1$$

□

Example 5.18 — Past Paper 2024 Q1(vii) & 2025 Q1(vi)

Show that matrix P is orthogonal if and only if P^T is orthogonal.

Solution: **($\Rightarrow$) Suppose P is orthogonal.** Then $P^TP = PP^T = I$. We must show $(P^T)^TP^T = P^T(P^T)^T = I$. Since $(P^T)^T = P$, this becomes:

$$(P^T)^TP^T = PP^T = I \quad \checkmark$$

$$P^T(P^T)^T = P^TP = I \quad \checkmark$$

So P^T is orthogonal.

($\Leftarrow$) Suppose P^T is orthogonal. Then by the same argument applied to P^T in place of P:

$$(P^T)^TP^T = I \implies PP^T = I \quad \checkmark$$
$$P^T(P^T)^T = I \implies P^TP = I \quad \checkmark$$

So P is orthogonal.

Since each condition implies the other, P is orthogonal if and only if P^T is orthogonal. □

Example 5.19 — Past Paper 2023 Q1(xi)

Find an orthogonal matrix whose first row is $\left(\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}\right)$.

Solution: For a 2×2 orthogonal matrix, the second row must be a unit vector orthogonal to the first row. A vector orthogonal to $\left(\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}\right)$ is obtained by swapping components and negating one: $\left(-\frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}}\right)$.

Check unit length: $\left(-\frac{2}{\sqrt{5}}\right)^2 + \left(\frac{1}{\sqrt{5}}\right)^2 = \frac{4}{5} + \frac{1}{5} = 1.$ ✓

Check orthogonality with row 1: $\frac{1}{\sqrt{5}}\left(-\frac{2}{\sqrt{5}}\right) + \frac{2}{\sqrt{5}}\left(\frac{1}{\sqrt{5}}\right) = -\frac{2}{5} + \frac{2}{5} = 0.$ ✓

$$P = \boxed{\begin{bmatrix} \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \\ -\frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{bmatrix}}$$

Properties of Orthogonal Matrices

1. $P^{-1} = P^T$ *(inverse equals transpose)*
2. $\det P = \pm 1$ *(determinant restriction)*
3. The rows of P form an orthonormal set, and so do the columns of P *(orthonormal rows and columns)*
4. P is orthogonal if and only if P^T is orthogonal *(transpose symmetry)*
5. The product of two orthogonal matrices is orthogonal *(closure under multiplication)*

Orthogonality — Exam Strategy:

1. To test orthogonality of two vectors, simply check whether $\langle u, v\rangle = 0$.
2. To find a vector orthogonal to two given vectors in $\mathbb{R}^3$, use the cross product.
3. To verify a matrix is orthogonal, check $P^TP = I$ (it is enough to check one of $P^TP = I$ or $PP^T = I$ for a square matrix).
4. Building an orthogonal matrix from a given row/column: each remaining row/column must be a unit vector orthogonal to all previously chosen rows/columns.

Orthogonal Sets and Bases

Orthogonal and Orthonormal Sets

A set $S = \{v_1, v_2, \ldots, v_n\}$ of **nonzero** vectors in an inner product space V is called an **orthogonal set** if every pair of distinct vectors in S is orthogonal:

$$\langle v_i, v_j\rangle = 0 \qquad \text{for all } i \neq j.$$

If, in addition, every vector in S has norm 1 (i.e. $\|v_i\| = 1$ for all i), then S is called an **orthonormal set**. Any orthogonal set can be converted into an orthonormal set by **normalizing** each vector (dividing it by its own norm).

Example 5.20 — Verifying an Orthogonal Set

Show that $S = \{(1,1,1),\ (1,-2,1),\ (1,0,-1)\}$ is an orthogonal set in $\mathbb{R}^3$.

Solution: Label the vectors $v_1 = (1,1,1)$, $v_2 = (1,-2,1)$, $v_3 = (1,0,-1)$, and check all three pairwise inner products:

$$\langle v_1, v_2\rangle = (1)(1) + (1)(-2) + (1)(1) = 1 - 2 + 1 = 0.$$

$$\langle v_1, v_3\rangle = (1)(1) + (1)(0) + (1)(-1) = 1 + 0 - 1 = 0.$$

$$\langle v_2, v_3\rangle = (1)(1) + (-2)(0) + (1)(-1) = 1 + 0 - 1 = 0.$$

Since all pairwise inner products are zero, S **is** an orthogonal set. □

Example 5.21 — Converting to an Orthonormal Set

Convert the orthogonal set from Example 5.20 into an orthonormal set.

Solution: Compute the norm of each vector:

$$\|v_1\| = \sqrt{1+1+1} = \sqrt{3}, \qquad \|v_2\| = \sqrt{1+4+1} = \sqrt{6}, \qquad \|v_3\| = \sqrt{1+0+1} = \sqrt{2}.$$

Divide each vector by its own norm:

$$q_1 = \frac{v_1}{\|v_1\|} = \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right), \qquad q_2 = \frac{v_2}{\|v_2\|} = \left(\frac{1}{\sqrt{6}}, \frac{-2}{\sqrt{6}}, \frac{1}{\sqrt{6}}\right),$$

$$q_3 = \frac{v_3}{\|v_3\|} = \left(\frac{1}{\sqrt{2}}, 0, \frac{-1}{\sqrt{2}}\right).$$

$$\{q_1, q_2, q_3\} = \boxed{\left\{\left(\tfrac{1}{\sqrt{3}}, \tfrac{1}{\sqrt{3}}, \tfrac{1}{\sqrt{3}}\right), \left(\tfrac{1}{\sqrt{6}}, \tfrac{-2}{\sqrt{6}}, \tfrac{1}{\sqrt{6}}\right), \left(\tfrac{1}{\sqrt{2}}, 0, \tfrac{-1}{\sqrt{2}}\right)\right\}}$$

Orthogonal Sets Are Linearly Independent

Every orthogonal set of nonzero vectors in an inner product space V is linearly independent.

Example 5.22 — Proving the Independence of Orthogonal Sets

Prove that an orthogonal set of nonzero vectors $\{v_1, v_2, \dots, v_n\}$ is linearly independent.

Solution: Suppose $a_1v_1 + a_2v_2 + \cdots + a_nv_n = 0$. Take the inner product of both sides with v_i (for a fixed but arbitrary index i):

$$\left\langle \sum_{j=1}^{n} a_j v_j,\ v_i \right\rangle = \langle 0, v_i\rangle = 0.$$

By linearity of the inner product:

$$\sum_{j=1}^{n} a_j \langle v_j, v_i\rangle = 0.$$

Since S is orthogonal, $\langle v_j, v_i\rangle = 0$ for every $j \neq i$, so all terms vanish except the $j = i$ term:

$$a_i\langle v_i, v_i\rangle = 0.$$

Since v_i is nonzero, $\langle v_i, v_i\rangle = \|v_i\|^2 \neq 0$. Therefore $a_i = 0$.

Since i was arbitrary, $a_1 = a_2 = \cdots = a_n = 0$, which is the trivial solution. Hence $\{v_1, \dots, v_n\}$ is **linearly independent**. □

Example 5.23 — Orthogonal Set as a Basis

Using the result of Example 5.22, explain why the orthogonal set $S = \{(1, 1, 1), (1, -2, 1), (1, 0, -1)\}$ from Example 5.20 is a basis of $\mathbb{R}^3$.

Solution: By Example 5.22, an orthogonal set of nonzero vectors is always linearly independent. Since S consists of 3 nonzero orthogonal vectors, S is linearly independent.

Since $\dim(\mathbb{R}^3) = 3$ and S contains exactly 3 independent vectors, S automatically spans $\mathbb{R}^3$ as well. Hence:

$$S \boxed{\text{is a basis of } \mathbb{R}^3}$$

(In fact, S is called an **orthogonal basis** of $\mathbb{R}^3$.) □

Orthogonal Basis and Fourier Coefficients

An **orthogonal basis** of a vector space V is a basis that is also an orthogonal set. If $S = \{v_1, v_2, \ldots, v_n\}$ is an orthogonal basis of V, every vector $v \in V$ can be written as:

$$v = c_1v_1 + c_2v_2 + \cdots + c_nv_n, \qquad \text{where } c_i = \frac{\langle v, v_i\rangle}{\langle v_i, v_i\rangle} = \frac{\langle v, v_i\rangle}{\|v_i\|^2}.$$

The scalars c_i are called the **Fourier coefficients** of v relative to S. If S is instead an **orthonormal** basis, the formula simplifies (since $\|v_i\| = 1$) to:

$$v = \langle v, v_1\rangle v_1 + \langle v, v_2\rangle v_2 + \cdots + \langle v, v_n\rangle v_n.$$

Example 5.24 — Fourier Coefficients Relative to an Orthogonal Basis

Express $v = (2, 3, -1)$ as a linear combination of the orthogonal basis $S = \{(1, 1, 1), (1, -2, 1), (1, 0, -1)\}$ using Fourier coefficients.

Solution: Compute each Fourier coefficient $c_i = \dfrac{\langle v, v_i\rangle}{\|v_i\|^2}$, reusing $\|v_1\|^2 = 3$, $\|v_2\|^2 = 6$, $\|v_3\|^2 = 2$ from Example 5.21:

$$\begin{aligned}
\langle v, v_1\rangle &= 2 + 3 - 1 = 4, & c_1 &= \frac{4}{3}.\\
\langle v, v_2\rangle &= 2 - 6 - 1 = -5, & c_2 &= \frac{-5}{6}.\\
\langle v, v_3\rangle &= 2 + 0 + 1 = 3, & c_3 &= \frac{3}{2}.
\end{aligned}$$

Verification:

$$\begin{aligned}
c_1v_1 + c_2v_2 + c_3v_3 &= \tfrac{4}{3}(1, 1, 1) + \left(-\tfrac{5}{6}\right)(1, -2, 1) + \tfrac{3}{2}(1, 0, -1)\\
&= \left(\tfrac{8}{6} - \tfrac{5}{6} + \tfrac{9}{6},\ \tfrac{8}{6} + \tfrac{10}{6} + 0,\ \tfrac{8}{6} - \tfrac{5}{6} - \tfrac{9}{6}\right)\\
&= \left(\tfrac{12}{6},\ \tfrac{18}{6},\ \tfrac{-6}{6}\right) = (2, 3, -1) = v. \quad \checkmark
\end{aligned}$$

$$v = \boxed{\tfrac{4}{3}v_1 - \tfrac{5}{6}v_2 + \tfrac{3}{2}v_3}$$

Example 5.25 — Coordinates Relative to an Orthonormal Basis

Express $v = (2, 3, -1)$ in terms of the orthonormal basis $\{q_1, q_2, q_3\}$ from Example 5.21.

Solution: For an orthonormal basis, the coefficients are simply $\langle v, q_i \rangle$ (no division by $\|v_i\|^2$ needed):

$$\langle v, q_1 \rangle = \frac{2+3-1}{\sqrt{3}} = \frac{4}{\sqrt{3}}.$$

$$\langle v, q_2 \rangle = \frac{2-6-1}{\sqrt{6}} = \frac{-5}{\sqrt{6}}.$$

$$\langle v, q_3 \rangle = \frac{2+0+1}{\sqrt{2}} = \frac{3}{\sqrt{2}}.$$

$$v = \boxed{\tfrac{4}{\sqrt{3}} q_1 - \tfrac{5}{\sqrt{6}} q_2 + \tfrac{3}{\sqrt{2}} q_3}$$

Notice these coefficients relate to the Fourier coefficients of Example 5.24 by $c_i' = c_i \|v_i\|$ — e.g. $c_1 \|v_1\| = \frac{4}{3} \cdot \sqrt{3} = \frac{4}{\sqrt{3}}$. ✓ □

Pythagorean Theorem

If u and v are orthogonal vectors in an inner product space V, then:

$$\|u+v\|^2 = \|u\|^2 + \|v\|^2.$$

Example 5.26 — Verifying the Pythagorean Theorem

Verify the Pythagorean Theorem for the orthogonal vectors $v_1 = (1, 1, 1)$ and $v_2 = (1, -2, 1)$ from Example 5.20.

Solution:

$$v_1 + v_2 = (2, -1, 2).$$

$$\|v_1 + v_2\|^2 = 2^2 + (-1)^2 + 2^2 = 4 + 1 + 4 = 9.$$

$$\|v_1\|^2 = 1 + 1 + 1 = 3, \qquad \|v_2\|^2 = 1 + 4 + 1 = 6.$$

$$\|v_1\|^2 + \|v_2\|^2 = 3 + 6 = 9.$$

Since $\|v_1 + v_2\|^2 = 9 = \|v_1\|^2 + \|v_2\|^2$, the Pythagorean Theorem is verified. □

Orthogonal Sets and Bases — Key Facts:

1. An orthogonal set of nonzero vectors is automatically linearly independent — no separate independence check is needed.
2. n nonzero orthogonal vectors in an n-dimensional space automatically form an orthogonal basis.
3. Fourier coefficients avoid solving a linear system: once a basis is known to be orthogonal, each coefficient is found directly via $c_i = \dfrac{\langle v, v_i \rangle}{\|v_i\|^2}$.
4. For an **orthonormal** basis, the formula simplifies further to $c_i = \langle v, v_i \rangle$, since $\|v_i\| = 1$.
5. The Pythagorean Theorem applies **only** when the two vectors are orthogonal — it is a quick way to verify orthogonality was used/computed correctly.

Gram-Schmidt Orthogonalization Process

Gram-Schmidt Orthogonalization Process

Let $\{u_1, u_2, \ldots, u_n\}$ be a basis of an inner product space V. The **Gram-Schmidt process** constructs an **orthogonal basis** $\{v_1, v_2, \ldots, v_n\}$ of V as follows:

$$v_1 = u_1,$$

$$v_2 = u_2 - \frac{\langle u_2, v_1 \rangle}{\langle v_1, v_1 \rangle} v_1,$$

$$v_3 = u_3 - \frac{\langle u_3, v_1 \rangle}{\langle v_1, v_1 \rangle} v_1 - \frac{\langle u_3, v_2 \rangle}{\langle v_2, v_2 \rangle} v_2,$$

and in general, for each $k = 1, \ldots, n$:

$$v_k = u_k - \sum_{i=1}^{k-1} \frac{\langle u_k, v_i \rangle}{\langle v_i, v_i \rangle} v_i.$$

Each new vector v_k is formed by subtracting from u_k its projections onto all previously constructed vectors $v_1, \ldots, v_{k-1}$. Normalizing each v_i (dividing by $\|v_i\|$) then produces an **orthonormal basis** $\{q_1, q_2, \ldots, q_n\}$ of V.

Existence of an Orthonormal Basis

Every finite-dimensional inner product space V has an orthogonal basis, and hence also an orthonormal basis. Both can be obtained by applying the Gram-Schmidt process to **any** basis of V.

Example 5.27 — Gram-Schmidt Process for Two Vectors

Apply the Gram-Schmidt process to $u_1 = (3, 4)$ and $u_2 = (1, 2)$ in $\mathbb{R}^2$ to find an orthogonal basis, then normalize to find an orthonormal basis.

Solution:

$$v_1 = u_1 = (3, 4).$$

$$\langle u_2, v_1 \rangle = (1)(3) + (2)(4) = 11, \qquad \langle v_1, v_1 \rangle = 9 + 16 = 25.$$

$$v_2 = u_2 - \frac{11}{25} v_1 = (1, 2) - \frac{11}{25}(3, 4) = \left(1 - \frac{33}{25},\ 2 - \frac{44}{25}\right) = \left(-\frac{8}{25},\ \frac{6}{25}\right).$$

Check: $\langle v_1, v_2 \rangle = 3\left(-\frac{8}{25}\right) + 4\left(\frac{6}{25}\right) = -\frac{24}{25} + \frac{24}{25} = 0.$ ✓

Normalize:

$$\|v_1\| = \sqrt{9 + 16} = 5, \qquad \|v_2\| = \sqrt{\tfrac{64}{625} + \tfrac{36}{625}} = \sqrt{\tfrac{100}{625}} = \frac{2}{5}.$$

$$q_1 = \frac{v_1}{\|v_1\|} = \left(\frac{3}{5}, \frac{4}{5}\right), \qquad q_2 = \frac{v_2}{\|v_2\|} = \left(-\frac{4}{5}, \frac{3}{5}\right).$$

$$\{q_1, q_2\} = \boxed{\left\{\left(\tfrac{3}{5}, \tfrac{4}{5}\right),\ \left(-\tfrac{4}{5}, \tfrac{3}{5}\right)\right\}}$$

Example 5.28 — Past Paper 2024 Q3(b) & 2025 Q3(b)

Apply the Gram-Schmidt process to the basis vectors $u_1 = (1, 1, 1)$, $u_2 = (0, 1, 1)$, $u_3 = (0, 0, 1)$ to transform them into an orthogonal basis, then normalize to obtain an orthonormal basis.

Solution:

$$v_1 = u_1 = (1, 1, 1), \qquad \langle v_1, v_1 \rangle = 3.$$

$$\langle u_2, v_1 \rangle = 0 + 1 + 1 = 2.$$

$$v_2 = u_2 - \frac{2}{3} v_1 = (0, 1, 1) - \tfrac{2}{3}(1, 1, 1) = \left(-\tfrac{2}{3},\ \tfrac{1}{3},\ \tfrac{1}{3}\right), \qquad \langle v_2, v_2 \rangle = \tfrac{4}{9} + \tfrac{1}{9} + \tfrac{1}{9} = \tfrac{2}{3}.$$

$$\langle u_3, v_1 \rangle = 0 + 0 + 1 = 1, \qquad \langle u_3, v_2 \rangle = 0 + 0 + \tfrac{1}{3} = \tfrac{1}{3}.$$

$$v_3 = u_3 - \frac{1}{3} v_1 - \frac{1/3}{2/3} v_2 = (0, 0, 1) - \tfrac{1}{3}(1, 1, 1) - \tfrac{1}{2}\left(-\tfrac{2}{3}, \tfrac{1}{3}, \tfrac{1}{3}\right)$$

$$= (0, 0, 1) - \left(\tfrac{1}{3}, \tfrac{1}{3}, \tfrac{1}{3}\right) - \left(-\tfrac{1}{3}, \tfrac{1}{6}, \tfrac{1}{6}\right) = \left(0,\ -\tfrac{1}{2},\ \tfrac{1}{2}\right).$$

Check: $\langle v_1, v_3 \rangle = 0 - \tfrac{1}{2} + \tfrac{1}{2} = 0$ ✓, $\langle v_2, v_3 \rangle = 0 - \tfrac{1}{6} + \tfrac{1}{6} = 0$ ✓

Normalize:

$$\|v_1\| = \sqrt{3}, \qquad \|v_2\| = \sqrt{\tfrac{2}{3}} = \tfrac{\sqrt{6}}{3}, \qquad \|v_3\| = \sqrt{\tfrac{1}{2}} = \tfrac{1}{\sqrt{2}}.$$

$$q_1 = \left(\tfrac{1}{\sqrt{3}}, \tfrac{1}{\sqrt{3}}, \tfrac{1}{\sqrt{3}}\right), \qquad q_2 = \left(-\tfrac{2}{\sqrt{6}}, \tfrac{1}{\sqrt{6}}, \tfrac{1}{\sqrt{6}}\right), \qquad q_3 = \left(0, -\tfrac{1}{\sqrt{2}}, \tfrac{1}{\sqrt{2}}\right).$$

$$\{q_1, q_2, q_3\} = \boxed{\left\{\left(\tfrac{1}{\sqrt{3}}, \tfrac{1}{\sqrt{3}}, \tfrac{1}{\sqrt{3}}\right), \left(-\tfrac{2}{\sqrt{6}}, \tfrac{1}{\sqrt{6}}, \tfrac{1}{\sqrt{6}}\right), \left(0, -\tfrac{1}{\sqrt{2}}, \tfrac{1}{\sqrt{2}}\right)\right\}}$$

Example 5.29 — Past Paper 2021 Q6

Show that $\{(1, -1, 0), (2, -1, -2), (1, -1, -2)\}$ is a basis of $\mathbb{R}^3$. Find an orthonormal basis of $\mathbb{R}^3$ using the Gram-Schmidt process.

Solution: Basis check: compute the determinant of the matrix with these vectors as rows:

$$\det \begin{bmatrix} 1 & -1 & 0 \\ 2 & -1 & -2 \\ 1 & -1 & -2 \end{bmatrix} = 1\big((-1)(-2) - (-2)(-1)\big) - (-1)\big((2)(-2) - (-2)(1)\big) + 0$$

$$= 1(2 - 2) + 1(-4 + 2) = 0 - 2 = -2.$$

Since $\det = -2 \neq 0$, the vectors are linearly independent and form a **basis** of $\mathbb{R}^3$.

Gram-Schmidt: let $u_1 = (1, -1, 0)$, $u_2 = (2, -1, -2)$, $u_3 = (1, -1, -2)$.

$$v_1 = u_1 = (1, -1, 0), \qquad \langle v_1, v_1 \rangle = 2.$$

$$\langle u_2, v_1 \rangle = 2 + 1 + 0 = 3.$$

$$v_2 = u_2 - \frac{3}{2} v_1 = (2, -1, -2) - \tfrac{3}{2}(1, -1, 0) = \left(\tfrac{1}{2},\ \tfrac{1}{2},\ -2\right), \qquad \langle v_2, v_2 \rangle = \tfrac{1}{4} + \tfrac{1}{4} + 4 = \tfrac{9}{2}.$$

$$\langle u_3, v_1 \rangle = 1 + 1 + 0 = 2, \qquad \langle u_3, v_2 \rangle = \tfrac{1}{2} - \tfrac{1}{2} + 4 = 4.$$

$$v_3 = u_3 - \frac{2}{2} v_1 - \frac{4}{9/2} v_2 = (1, -1, -2) - (1, -1, 0) - \tfrac{8}{9}\left(\tfrac{1}{2}, \tfrac{1}{2}, -2\right)$$

$$= (0, 0, -2) - \left(\tfrac{4}{9}, \tfrac{4}{9}, -\tfrac{16}{9}\right) = \left(-\tfrac{4}{9},\ -\tfrac{4}{9},\ -\tfrac{2}{9}\right).$$

Normalize:

$$\|v_1\| = \sqrt{2}, \qquad \|v_2\| = \sqrt{\tfrac{9}{2}} = \tfrac{3}{\sqrt{2}}, \qquad \|v_3\| = \sqrt{\tfrac{16}{81} + \tfrac{16}{81} + \tfrac{4}{81}} = \sqrt{\tfrac{36}{81}} = \tfrac{2}{3}.$$

$$q_1 = \left(\tfrac{1}{\sqrt{2}}, -\tfrac{1}{\sqrt{2}}, 0\right), \qquad q_2 = \left(\tfrac{\sqrt{2}}{6}, \tfrac{\sqrt{2}}{6}, -\tfrac{2\sqrt{2}}{3}\right), \qquad q_3 = \left(-\tfrac{2}{3}, -\tfrac{2}{3}, -\tfrac{1}{3}\right).$$

$$\{q_1, q_2, q_3\} = \boxed{\left\{\left(\tfrac{1}{\sqrt{2}}, -\tfrac{1}{\sqrt{2}}, 0\right), \left(\tfrac{\sqrt{2}}{6}, \tfrac{\sqrt{2}}{6}, -\tfrac{2\sqrt{2}}{3}\right), \left(-\tfrac{2}{3}, -\tfrac{2}{3}, -\tfrac{1}{3}\right)\right\}}$$

Example 5.30 — Past Paper 2023 Q6

Show that $\{(1,0,1), (0,1,1), (0,0,1)\}$ is a basis of $\mathbb{R}^3$. Find an orthonormal basis of $\mathbb{R}^3$ using the Gram-Schmidt process.

Solution: Basis check: the matrix with these vectors as rows is upper triangular:

$$\det \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} = (1)(1)(1) = 1 \neq 0.$$

Since the determinant is nonzero, the vectors are linearly independent and form a **basis** of $\mathbb{R}^3$.

Gram-Schmidt: let $u_1 = (1,0,1)$, $u_2 = (0,1,1)$, $u_3 = (0,0,1)$.

$$v_1 = u_1 = (1,0,1), \qquad \langle v_1, v_1\rangle = 2.$$

$$\langle u_2, v_1\rangle = 0 + 0 + 1 = 1.$$

$$v_2 = u_2 - \frac{1}{2}v_1 = (0,1,1) - \tfrac{1}{2}(1,0,1) = \left(-\tfrac{1}{2},\ 1,\ \tfrac{1}{2}\right), \qquad \langle v_2, v_2\rangle = \tfrac{1}{4} + 1 + \tfrac{1}{4} = \tfrac{3}{2}.$$

$$\langle u_3, v_1\rangle = 0 + 0 + 1 = 1, \qquad \langle u_3, v_2\rangle = 0 + 0 + \tfrac{1}{2} = \tfrac{1}{2}.$$

$$v_3 = u_3 - \frac{1}{2}v_1 - \frac{1/2}{3/2}v_2 = (0,0,1) - \tfrac{1}{2}(1,0,1) - \tfrac{1}{3}\left(-\tfrac{1}{2}, 1, \tfrac{1}{2}\right)$$
$$= \left(-\tfrac{1}{2}, 0, \tfrac{1}{2}\right) - \left(-\tfrac{1}{6}, \tfrac{1}{3}, \tfrac{1}{6}\right) = \left(-\tfrac{1}{3},\ -\tfrac{1}{3},\ \tfrac{1}{3}\right).$$

Normalize:

$$\|v_1\| = \sqrt{2}, \qquad \|v_2\| = \sqrt{\tfrac{3}{2}} = \tfrac{\sqrt{6}}{2}, \qquad \|v_3\| = \sqrt{\tfrac{1}{9} + \tfrac{1}{9} + \tfrac{1}{9}} = \tfrac{1}{\sqrt{3}}.$$

$$q_1 = \left(\tfrac{1}{\sqrt{2}}, 0, \tfrac{1}{\sqrt{2}}\right), \qquad q_2 = \left(-\tfrac{1}{\sqrt{6}}, \tfrac{2}{\sqrt{6}}, \tfrac{1}{\sqrt{6}}\right), \qquad q_3 = \left(-\tfrac{1}{\sqrt{3}}, -\tfrac{1}{\sqrt{3}}, \tfrac{1}{\sqrt{3}}\right).$$

$$\{q_1, q_2, q_3\} = \boxed{\left\{\left(\tfrac{1}{\sqrt{2}}, 0, \tfrac{1}{\sqrt{2}}\right), \left(-\tfrac{1}{\sqrt{6}}, \tfrac{2}{\sqrt{6}}, \tfrac{1}{\sqrt{6}}\right), \left(-\tfrac{1}{\sqrt{3}}, -\tfrac{1}{\sqrt{3}}, \tfrac{1}{\sqrt{3}}\right)\right\}}$$

Gram-Schmidt — Exam Strategy:

1. Always compute $v_1 = u_1$ first, then find v_2 using only v_1, then v_3 using **both** v_1 and v_2 — never skip a projection term.
2. Keep $\langle v_i, v_i\rangle$ values recorded as you go; they are reused repeatedly in later projection coefficients.
3. Always verify orthogonality of each new v_k against all previous v_i before moving on — this catches arithmetic errors early.
4. Normalize **only at the very end**, after the full orthogonal basis $\{v_1, \ldots, v_n\}$ has been

found — normalizing early makes the fractions in later projection steps far messier.

5. If the problem also asks to verify the given vectors form a basis first, use the determinant test before starting Gram-Schmidt.

Practice Problems

1. Let $u = (2, -1, 4)$ and $v = (3, 0, -2)$ in $\mathbb{R}^3$. Compute $\langle u, v\rangle$ and $\|u\|$.

2. Compute $\langle u, v\rangle$ for $u = (1, -2)$ and $v = (3, 1)$ using the weighted inner product with weights $w_1 = 3,\ w_2 = 2$.

3. Let $u = (1 + i,\ 2)$ and $v = (2 - i,\ 1 + i)$ in $\mathbb{C}^2$. Compute $\langle u, v\rangle$.

4. Let $f(t) = 1$ and $g(t) = t^2$ in $C[0, 2]$. Compute $\langle f, g\rangle = \int_0^2 f(t)g(t)\,dt$.

5. Normalize the vector $v = (2, -3, 6)$.

6. Verify the Cauchy-Schwarz Inequality for $u = (2, 1, -1)$ and $v = (1, 3, 2)$.

7. Verify the Triangle Inequality for $u = (1, -1, 2)$ and $v = (3, 0, -1)$.

8. Find the angle between $u = (1, 0, -1)$ and $v = (2, 1, 2)$.

9. Determine whether $u = (3, 2, -1)$ and $v = (1, -2, -1)$ are orthogonal.

10. Find $W^\perp$ where $W = \text{span}\{(1, 2, -1)\}$ in $\mathbb{R}^3$.

11. Find a vector orthogonal to $u_1 = (1, 0, 2)$ and $u_2 = (2, 1, -1)$.

12. Show that $P = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ is an orthogonal matrix.

13. If A is a 3×3 orthogonal matrix with $\det A = -1$, find $\det(A^2)$.

14. Find an orthogonal matrix whose first row is $\left(\frac{3}{5}, \frac{4}{5}\right)$.

15. Determine whether $\{(1, 2, 1), (2, -1, 0), (1, 2, -5)\}$ is an orthogonal set in $\mathbb{R}^3$.

16. Convert the orthogonal set from the previous problem into an orthonormal set.

17. Explain why an orthogonal set of 4 nonzero vectors in $\mathbb{R}^4$ must automatically be a basis of $\mathbb{R}^4$.

18. Express $v = (1, 4, -2)$ as a linear combination of the orthogonal basis $\{(1, 1, 0), (1, -1, 0), (0, 0, 1)\}$ using Fourier coefficients.

19. Verify the Pythagorean Theorem for the orthogonal vectors $u = (2, 0, -2)$ and $v = (1, 3, 1)$.

20. Apply the Gram-Schmidt process to $u_1 = (1, 1, 0)$ and $u_2 = (2, 0, 1)$ in $\mathbb{R}^3$ to find an orthogonal basis.

21. Apply the Gram-Schmidt process to $u_1 = (1, 1, 1)$, $u_2 = (1, 0, 1)$, $u_3 = (1, 1, 0)$ to find an orthonormal basis of $\mathbb{R}^3$.

22. Show that $\{(1,2,0),(0,1,1),(1,0,1)\}$ is a basis of $\mathbb{R}^3$, then apply the Gram-Schmidt process to find an orthonormal basis.

23. Apply the Gram-Schmidt process to $u_1 = (1,0,1,0)$, $u_2 = (1,1,0,0)$, $u_3 = (0,1,1,1)$ in $\mathbb{R}^4$ to obtain an orthogonal set.

24. Find an orthonormal basis for the subspace $W = \text{span}\{(1,1,0),(0,1,1)\}$ of $\mathbb{R}^3$ using the Gram-Schmidt process.

25. Let $W = \text{span}\{(1,-1,1)\}$ in $\mathbb{R}^3$. Find a spanning set for $W^\perp$, then apply the Gram-Schmidt process to obtain an orthonormal basis of $W^\perp$.